Simple Algebraic Treatment of Unsteady Heat Conduction in Solid Spheres with Heat Convection Exchange to Fluids Covering the Entire Time Domain

Marcelo Marucho¹* and Antonio Campo-Garcia²

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Abstract

Accurate estimates of temperature histories in solid bodies of regular shape (slabs, cylinders and spheres) exposed to cold fluids have been traditionally done by the method of separation of variables evaluating infinite Fourier series. The infinite Fourier series correspond to the exact, analytic solution of the unidirectional heat conduction equation in various coordinate systems. The central goal of this technical paper is to bypass this traditional procedure involving infinite Fourier series. The idea is to predict the three most important temperatures (mean, surface and center) in a solid sphere by implementing a new 1-D composite lumped analysis, which constitutes a natural extension of the standard 1-D lumped analysis. The computational methodology to be proposed is effortless and brings to the table a handful of compact algebraic equations for the mean, surface and center temperatures that cover the complete gamma of the controlling Biot numbers (0 < Bi < 100) in the entire time domain, i.e., at all dimensionless times or Fourier numbers (0 < τ < ∞).

Keywords: Unsteady 1-D heat conduction; Solid sphere; 1-D composite lumped model; Temperature distribution; Total heat transfer

Nomenclature

\[ A_s \] surface area
\[ Bi \] Biot number, \[ \frac{\overline{h_f} R}{k_s} \]
\[ Bi_l \] lumped Biot number, \[ \frac{\overline{h_f}}{k_s} \left( \frac{V}{A_s} \right) \]
\[ D \] diameter

*Corresponding e-mail: marcelo.marucho@utsa.edu
1 Department of Physics, The University of Texas at San Antonio, San Antonio, TX 78249
2 Department of Mechanical Engineering, The University of Texas at San Antonio, San Antonio, TX 78249
specific heat capacity of solid
mean, external convective coefficient
mean, "internal convective" coefficient
thermal conductivity of fluid
thermal conductivity of solid
mean, internal Nusselt number, \( \frac{\bar{h}_s D}{k_s} \)
mean, external Nusselt number, \( \frac{\bar{h}_f D}{k_f} \)
mean, overall Nusselt number, \( \frac{\bar{U}D}{k_s} \)
Prandtl number in eqs. (A.10) and (A.11)
radial coordinate
radius
Rayleigh number in eq. (A.11)
external convective resistance
Reynolds number in eq. (A.10)
time
temperature
center temperature
fluid temperature
initial temperature
mean temperature
surface temperature
mean, overall heat transfer coefficient
volume
time-dependent coefficients in eq. (19)
thermal diffusivity of solid, \( \frac{k_s}{\rho_s c_s} \)
dimensionless radial coordinate, \( \frac{r}{R} \)
dimensionless excess temperature \( T - T_f \) for convective surface, \( \frac{T - T_f}{T_{in} - T_f} \)
dimensionless excess temperature \( T - T_f \) for constant temperature surface, \( \frac{T - T_w}{T_{in} - T_w} \)
density of solid
dimensionless time or Fourier number, \( \frac{t}{R^2/\alpha_s} \)
initial
fluid
final
overall
solid
surface
Greek letters
Subscripts
1. Introduction

The temperature of hot bodies immersed in cold fluids in general varies with both position and time. The heat treatment process is called quenching (Liscic et al. [1]). In general, the exact analytical solutions of the heat conduction equation for simple bodies (large slabs, long cylinders and spheres) consists of Fourier infinite series with input from with transcendental equations framed in Cartesian, cylindrical and spherical coordinates, which are available in heat conduction books, such as Carslaw and Jaeger [2], Arpaci [3], Luikov [4], Grigull and Sanders [5], Ozisik [6], Poulilakos [7] and Myers [8]. Herein, the local temperature is a double function of position and time influenced by the mean, external convective coefficient associated with the fluid.

The central objective of this technical paper is to develop a set of algebraic correlation equations of compact form that facilitate quick, direct and accurate estimations of the three most important temperatures (mean, surface and center) in a solid sphere, while covering the entire time domain \(0 < \tau < \infty\). The sphere is chosen here because it has the smallest surface area among all surfaces enclosing a given volume or alternatively the sphere encloses the largest volume among all closed surfaces with a given surface area.

The main objective of the present technical paper is to devise a new 1-D composite lumped model revolving around the mean, overall heat transfer coefficient, \(\bar{U}\), linking the heat by conduction inside the solid sphere and the heat by convection between the sphere surface and the surrounding fluid in which it is immersed. \(\bar{U}\) contains two quantities: one is an artificial mean, “internal convective coefficient”, \(\bar{h}_s\), varying with time \(t\) and the other is the natural mean, external convective coefficient, \(\bar{h}_f\), invariant with time \(t\).

The technical paper is divided in four parts. The first part deals with the formulation of the 1-D composite lumped model accounting for two thermal resistances: the artificial “internal conduction resistance” and the natural external convective resistance. The pair of dimensionless mean and surface temperatures is developed in the second part. In the third part, the dimensionless center temperatures are obtained indirectly by way of a trial dimensionless temperature profile of polynomial structure that needed to be formulated. A representative collection of numerical values for the dimensionless center, surface and mean temperatures are presented in the fourth part for various cooling/heating conditions. The three numerical dimensionless temperatures are compared against the exact dimensionless temperatures evaluated from the Fourier infinite series.

2. Distributed Model

Consider unsteady, unidirectional heat conduction in a solid body in the form of a sphere shown in Figure 1. The solid sphere is maintained initially at a temperature \(T_m\). At time \(t = 0\), the sphere’s surface is suddenly exposed to a cold fluid (liquid, gas or vapor). The heat convection is characterized by a fluid temperature, \(T_f\) and a mean, external convective coefficient, \(\bar{h}_f\). Accordingly, the 1-D heat conduction equation framed in spherical coordinates [2-8] is

\[
\frac{1}{\alpha_s} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \tag{1}
\]
where $\alpha$ is the thermal diffusivity of the solid. The initial condition is

$$T(r, 0) = T_{in}$$  \hspace{1cm} (2)

and the boundary conditions are

$$\frac{\partial T(0, t)}{\partial r} = 0$$ \hspace{1cm} (3a)

$$-k_s \frac{\partial T(R, t)}{\partial r} = h_f [T(R, t) - T_f]$$ \hspace{1cm} (3b)

Adopting the customary dimensionless variables $\theta$ for the excess temperature $T - T_f$, $\eta$ for the radial coordinate $r$ and $\tau$ for the time $t$,

$$\theta = \frac{T - T_f}{T_{in} - T_f}, \quad \eta = \frac{r}{R}, \quad \tau = \frac{t}{R^2/\alpha_s}$$ \hspace{1cm} (4)

the Biot number $Bi = \frac{h_f R}{k_s}$ surfaces up from equation (3b). This is a dimensionless parameter that controls the unsteady heat conduction problem. Physically speaking, $Bi$ represents the ratio of the internal conductive resistance within the solid sphere, $R_k$, to the external convective resistance at the sphere surface, $R_c$.

$$Bi = \frac{R_k}{R_c}$$

In theory, the magnitude of $Bi$ ranges between zero and infinity, but in practice $Bi$ is restricted between zero and one hundred (Mills [9]).

The exact, analytic 1-D dimensionless temperature distribution in a solid sphere, $\theta(\eta, \tau)$, as found in [2-8] is

$$\theta(\eta, \tau) = 2 \sum_{n=1}^{\infty} \left[ \frac{\sin(\mu_n) - \mu_n \cos(\mu_n)}{\mu_n - \sin(\mu_n) \cos(\mu_n)} \right] \exp \left( -\mu_n^2 \tau \right) \left[ \frac{\sin(\mu_n \eta)}{\mu_n \eta} \right]$$ \hspace{1cm} (5)

where the eigenvalues $\mu_n$ are the positive roots of the transcendental equation

$$\mu_n \cos (\mu_n) = (1 - Bi) \sin (\mu_n), \quad n = 1, 2, 3, ....$$ \hspace{1cm} (5a)
The pressing characteristic of the Fourier infinite series in eq. (5) revolves around convergence [9]. On one hand, the Fourier infinite series converge rapidly for long times \((\tau \gg 0)\), and generally one term is only needed. On the other hand, the Fourier infinite series diverge markedly for short times \((\tau \to 0)\), and numerous terms must be retained to guarantee good precision. The threshold time that separates the long time region from the short time region is a so-called critical dimensionless time \(\tau_{cr} = 0.2\).

From theoretical concepts, the contribution of the first positive root \(\mu_1\) in the transcendental equation (5a) toward the dimensionless temperature calculations in eq. (5) must be dominant after the critical dimensionless time \(\tau_{cr} = 0.2\) is surpassed. In other words, the implication is that the contribution of the successive positive roots \(\mu_2, \mu_3, \ldots, \mu_n\) in eqs. (5a) may be taken as negligible. Consequently, the infinite series in eq. (5) may be approximated by the truncated "one-term" series,

\[
\theta_1(\eta, t) = 2 \left[ \frac{\sin(\mu_1) - \mu_1 \cos(\mu_1)}{\mu_1 - \sin(\mu_1) \cos(\mu_1)} \right] \exp \left( -\mu_1^2 \tau \right) \left[ \frac{\sin(\mu_1 \eta)}{\mu_1 \eta} \right]
\]

(6)

where the first eigenvalue \(\mu_1\) is the positive roots of the transcendental equation

\[
\mu_1 \cos(\mu_1) = (1 - Bi) \sin(\mu_1), \quad n = 1, 2, 3, \ldots
\]

(6a)

### 3. Simple Lumped Model

#### 3.1 Mean temperature

From the mean value theorem [2-8], the mean temperature \(T_m\) in a volume \(V\) is defined by

\[
T_m(t) = \frac{1}{V} \int_V T(r, t) \, dV
\]

(7)

where \(T(r, t)\) is the temperature distribution. Let us define an infinitesimal control volume of thickness \(\Delta t\) in a solid body having mean temperature \(T_m\). A thermodynamic energy balance delivers the 1-D lumped heat equation along with the initial condition

\[
\rho_s c_s V \frac{dT_m}{dt} = - \overline{h_f} A_s (T_m - T_f), \quad T_m(0) = T_{in}
\]

(8)

where \(A_s\) is the surface area and \(V\) is the volume of the solid body. In the case of a solid sphere, this equation becomes

\[
\rho_s c_s R \frac{dT_m}{dt} = - 3 \overline{h_f} (T_m - T_f), \quad T_m(0) = T_{in}
\]

(8a)

where for this configuration

\[
T_m(t) = 3 \int_0^R T(r, t) \, r^2 \, dr
\]

(7a)
The solution of the first-order ordinary differential equation (8a) gives rise to the mean temperature distribution:

$$\frac{T_m - T_f}{T_{in} - T_f} = \exp \left( -\frac{3\overline{h}_f}{\rho_s c_s R} t \right)$$

(9)

Herein, the constraint imposed on eq. (9) corresponds to the lumped Biot number,

$$Bi_l = \frac{\overline{h}_s V}{k_s A_s} = \frac{\overline{h}_f R}{k_s (R/3)} < 0.1$$

where the subscript "l" in Bi indicates lumped.

4. New Composite Lumped Model

4.1 Mean temperature and surface temperature

Under the platform of a composite lumped model, an infinitesimal control volume of thickness $\Delta t$ in a solid body with mean temperature $T_m$ and surface temperature, $T_w$ is now constructed. A thermodynamic energy balance delivers the 1-D composite lumped heat equation along with the initial condition

$$\rho_s c_s V \frac{dT_m}{dt} = -U A_s (T_m - T_f), \quad T_m(0) = T_{in}$$

(10)

where $V$ is the volume and $A_s$ is the surface area of the solid body. In the case of a solid sphere, the above equation reduces to

$$\rho_s c_s R \frac{dT_m}{dt} = -3U (T_m - T_f), \quad T_m(0) = T_{in}$$

(10a)

The cornerstone of the new 1-D composite lumped model is the mean, overall heat transfer coefficient, $\overline{U}$, linking the heat by conduction inside the solid sphere and the heat by convection between the sphere surface and the surrounding fluid in which it is immersed. $\overline{U}$ contains two quantities: one is an artificial mean, “internal convective coefficient”, $\overline{h}_s$, varying with time $t$ and the other is the natural mean, external convective coefficient, $\overline{h}_f$, invariant with time $t$. That is,

$$\frac{1}{\overline{U}} = \frac{1}{\overline{h}_s} + \frac{1}{\overline{h}_f}$$

(11)

The structure of this equation indicates that $\overline{h}_s$ and $\overline{h}_f$ act in parallel in an equivalent electric circuit [9].

Within thermophysics, the 1-D composite lumped heat model seeks to articulate two fundamental sub-problems. One fundamental sub-problem envisions unsteady heat conduction in a solid sphere with prescribed surface temperature, $T_w$. Herein, a new concept of the artificial mean, “internal convective coefficient”, $\overline{h}_s$, varying with time $t$ has to be introduced, whose numerical values
emanate from the spatio-temporal temperature distribution in the sphere with constant surface temperature, \( T(r, t) \). The other fundamental sub-problem is connected to steady external convection between the solid sphere with uniform temperature, \( T_w \), and the adjacent fluid at a free-stream temperature \( T_f \) wherein \( \overline{h_f} \) is the mean, external convective coefficient. Normally, \( \overline{h_f} \) is obtained from standard correlation equations; this issue will be addressed later.

Next, multiplying the denominators in equation (11) by \( \frac{D}{k_s} \) supplies the dimensionless mean, overall heat transfer coefficient \( \overline{U_D} \), which is called the mean, overall Nusselt number, \( \overline{Nu_{ov}} \). In equation form, that is

\[
\frac{1}{\overline{Nu_{ov}}} = \frac{1}{\overline{Nu_s}} + \frac{1}{\left(\frac{k_f}{k_s}\right)} \overline{Nu_f}
\]

In the analysis of unsteady heat conduction, the dimensionless mean, external convective coefficient, \( \overline{h_f} \), is traditionally named the Biot number, \( Bi \) [2-8]. Consequently, the interconnection between \( Bi \) and \( \overline{Nu_s} \) boils down to

\[
Bi = \frac{1}{2} \left(\frac{k_f}{k_s}\right) \overline{Nu_f}
\]

Qualitatively, half of the fluid-to-solid thermal conductivity ratio \( \frac{1}{2} \left(\frac{k_f}{k_s}\right) \) supplies the constant of proportionality. In view of the foregoing, equation (12) may be conveniently re-expressed now in terms of \( Bi \) to become

\[
\frac{1}{\overline{Nu_{ov}}} = \frac{1}{\overline{Nu_s}} + \frac{1}{2Bi}
\]

For convenience, the mean excess temperature \( T_m - T_f \) and the time \( t \) are properly defined in dimensionless form as

\[
\theta_m = \frac{T_m - T_f}{T_{in} - T_f'} \quad \tau = \frac{t}{R^2/\alpha_s}
\]

Combining equations (10a), (14) and (15) generates the equivalent dimensionless 1-D composite lumped heat equation along with the initial condition

\[
\frac{d\theta_m}{d\tau} = -\frac{3}{2} \overline{Nu_{ov}} \theta_m \quad \theta_m(0) = 1
\]

Integration of equation (16) gives the dimensionless mean temperature distribution in the sphere (global quantity):

\[
\theta_m = \exp \left( -\frac{3}{2} \overline{Nu_{ov}} \tau \right)
\]
where the single dimensionless number $\bar{Nu}_{ov}$ controls the conductive-convective heat exchange between the solid sphere and the cold fluid at any given dimensionless time $\tau$.

At this stage, it is important to recognize that the derivation of the 1-D composite lumped heat equation (10a) does not pose any time restriction, signifying that its solution, the exponential dimensionless temperature distribution in equation (17) is valid for all dimensionless time, $0 < \tau < \infty$.

The second temperature of importance in the sphere is the surface temperature, $T_w$. The dimensionless surface temperature, $\theta_w = \frac{T_w - T_f}{T_{in} - T_f}$ (a local quantity) may be easily extracted from a combination of two of the three participating thermal resistances. For instance, one possible choice is pairing $\bar{Nu}_s$ with $\bar{Nu}_{ov}$, resulting in

$$\theta_w = \theta_m \left( 1 - \frac{\bar{Nu}_{ov}}{\bar{Nu}_s} \right) \quad (18)$$

4.2 Off-surface temperatures

The third temperature of importance in the sphere is the center temperature, $T_c$ (a local quantity), whose dimensionless form is $\theta_c = \frac{T_c - T_f}{T_{in} - T_f}$. Unfortunately, the 1-D composite lumped analysis developed here is unable to handle $\theta_c$. Instead, a special physics-based algebraic procedure has to be implemented. The assumed temperature distribution can be any arbitrary function provided that the boundary conditions are satisfied[10]. Therefore, let us propose the following trial dimensionless temperature of polynomial form

$$\theta(\eta, \tau) = \alpha(\tau)(1 - \eta^4) + \beta(\tau)(1 - \eta^2) + \gamma(\tau) \quad (19)$$

where $\alpha(\tau)$, $\beta(\tau)$ and $\gamma(\tau)$ are embedded coefficients that depend upon the dimensionless time $\tau$.

Correspondingly, the dimensionless temperature derivative $\theta'(\eta, \tau)$ is

$$\theta'(\eta, \tau) = -4\alpha(\tau) \eta^3 - 2\beta(\tau) \eta \quad (20)$$

To comply with the physics of the problem, the trial dimensionless temperature $\theta(\eta, \tau)$ must satisfy two conditions:

1) at the sphere’s center $\eta = 0$, the temperature derivative (due to symmetry) must be zero, i.e., $\theta'(0) = 0$, and
2) at the sphere’s surface $\eta = 1$, the coefficient $\gamma(\tau)$ matches the surface temperature, i.e., $\theta(1) = \theta_w$.

As a direct consequence, the trial dimensionless temperature simplifies to

$$\theta(\eta, \tau) = \alpha(\tau)(1 - \eta^4) + \beta(\tau)(1 - \eta^2) + \theta_w \quad (21)$$

where $\theta_w$ is taken from equation (18).
From now on, we prefer to write \( \alpha(\tau) = \alpha \) and \( \beta(\tau) = \beta \) for conciseness.

Performing a local energy balance at the dimensionless surface of the sphere \( \eta = 1 \), delivers an equation connecting the dimensionless temperature \( \theta \) and the dimensionless temperature derivative \( \theta' \) at this extreme location, \( \eta = 1 \). In other words, this step provides the linear algebraic equation

\[
\theta'_w = -Bi \theta_w \tag{22}
\]

In addition, evaluating the dimensionless temperature derivative \( \theta'(1, \tau) \) in equation (20) dispenses the relation

\[
\theta'_w = -4\alpha(\tau) - 2\beta(\tau) \tag{23}
\]

Once the values of \( \theta_m \) from equation (17) and \( \theta_w \) from equation (18) are quantified from the 1-D composite lumped model, the elementary algebraic procedure furnishes the following system of two algebraic equations

\[
Bi \theta_w = 4\alpha + 2\beta \tag{24a}
\]
\[
\theta_m = 3 \int_0^1 \theta \eta^2 \, d\eta = f(\alpha, \beta, \theta_w) \tag{24b}
\]

where the unknown variables are the embedded coefficients \( \alpha \) and \( \beta \).

5. Applications for Various Cooling/Heating Conditions

Introducing equation (21) into equation (24b) and carrying out the integration leads to the expression

\[
\theta_m = \frac{4}{7} \alpha + \frac{2}{5} \beta + \theta_w \tag{25}
\]

where the coefficients \( \alpha \) and \( \beta \) come from solving the system of equations (24a) and (24b). At the end, the resulting expressions for evaluating \( \alpha \) and \( \beta \) are

\[
\alpha = -\frac{35}{8} \theta_m + \frac{1}{5}(7Bi + 35)\theta_w \tag{26}
\]

and

\[
\beta = \frac{35}{4} \theta_m - \frac{5}{4}(Bi + 7)\theta_w \tag{27}
\]

Further, evaluating \( \theta(0, \tau) \) in equation (21) conveys the dimensionless center temperature

\[
\theta_c = \frac{35}{8} \theta_m - \frac{1}{8}(3Bi + 27)\theta_w \tag{28}
\]

where \( \theta_m \) is obtained from equation (17) and \( \theta_w \) from equation (18), respectively.
It should be emphasized that the proposed trial dimensionless temperature in equation (21) is versatile. Despite that other dimensionless local temperatures in the solid sphere (different than the center and surface) are of lesser importance, if needed, they can be computed with equation (21) as well. For example, the mid-point dimensionless temperature \(\theta \left( \frac{1}{2} \right)\) in the sphere at any dimensionless time \(\tau\) is computed from equation (21), resulting in

\[
\theta \left( \frac{1}{2} \right) = \frac{15}{16} \alpha + \frac{3}{4} \beta + \theta_w \tag{29}
\]

**6. Sequence of Algebraic Calculations**

Once the Biot number, \(Bi\), and the dimensionless time or Fourier number, \(\tau\), are known, the sequence of algebraic calculations may be summarized as follows:

1. Evaluate \(\overline{Nu_s}\) from equation (A.9) in the Appendix
2. Evaluate \(\overline{Nu_{ov}}\) from equation (14)
3. Calculate \(\theta_m\) from equation (17)
4. Calculate \(\theta_o\) from equation (18)
5. Calculate \(\theta_c\) from equation (28)

**7. Presentation and Comparison of Results**

The goodness of the three dimensionless temperatures predicted algebraically by the combination of the 1-D composite lumped heat equation and the physics-based procedure is assessed in detailed form in this section. In this regard, the available baseline solution is the spatio-temporal temperature distribution in equation (5) retaining between 1 and 1,000 terms to warrant convergence was evaluated with Mathematica [11]. It should be realized that the dimensionless time sub-region \(\tau < 0.2\) constitutes a special condition because the Fourier infinite series of equation (5) necessitates a large number of terms for reasonable convergence.

The required algebraic calculations will be done for three possible cases: 1) vigorous heat exchange, 2) moderate heat exchange and 3) weak heat exchange. The first step begins with the calculation of the dimensionless mean temperature, \(\theta_m\) (a global quantity) with equation (17). The second step continues with the calculation of the dimensionless surface temperatures, \(\theta_o\) (a local quantity) with equation (18). The third step culminates with the calculation of the dimensionless center temperature, \(\theta_c\) (a local quantity), with equation (28).

First, for the case of vigorous heat exchange, the internal conductive resistance \(R_i\) dominates the external convective resistance \(R_c\) (indicative of large \(Bi\)). Choosing a value of \(Bi = 10\), the dimensionless surface temperature, \(\theta_o\), drops drastically from 1 to 0.1 in a short dimensionless time \(\tau = 0.1\). This situation constitutes a crucial test for the 1-D composite lumped model united with the physics-based algebraic procedure developed here; this turns out to be quite interesting. Accordingly, at \(\tau = 0.1\) the mean, internal Nusselt number, \(\overline{Nu_s} = 9.88\) from equation (A.9). Introducing this number along with \(Bi = 10\) in equation (14) delivers the mean, overall Nusselt number, \(\overline{Nu_{ov}} = 6.61\). When these numbers are inserted in equations (17) and (18), the outcome is...
\( \theta_m = 0.37 \) and \( \theta_w = 0.12 \). The exact \( \theta_m = 0.35 \) and \( \theta_w = 0.10 \) evaluated from equation (5) shows a small deviation of 0.02 units with respect to the present calculation. Calculation of the center temperature, \( \theta_c \), from equation (28) supplies \( \theta_c = 0.76 \) as opposed to the exact 0.78 from equation (5); that is a small deviation of 0.02 units. Certainly, the agreement for the trio of dimensionless temperatures \( \theta_m, \theta_w, \theta_c \) is excellent.

Second, for the case of moderate heat exchange characterized by an internal conductive resistance comparable to the external convective resistance, in other words \( R_i \approx R_o \). This intermediate region may be adequately represented by a combination of \( Bi = 0.5 \) and \( \tau = 1.4 \). For the relative large time \( \tau = 1.4 \), the mean, internal Nusselt number from equation (A.9) gives \( \bar{Nu}_s = 7.07 \). Introducing this number along with \( Bi = 0.5 \) in equation (14) supplies the mean, overall Nusselt number \( \bar{Nu}_{ov} = 0.88 \). Next, using equations (17) and (18), the two influential dimensionless temperatures furnished by the 1-D composite lumped model produce \( \theta_m = 0.16 \) and \( \theta_w = 0.14 \). In addition, employing equation (28) related to the physics-based procedure, yields the dimensionless temperature \( \theta_c = 0.20 \). On the other hand, evaluating equation (5) gives \( \theta_m = 0.135 \) and \( \theta_w = 0.18 \). Then, the deviation for \( \theta_m \) is 0.005 units, whereas the deviation for \( \theta_c \) is 0.02 units.

Third, for the case of weak heat exchange, the external convective resistance overshadows the internal conductive resistance, \( R_i >> R_o \) (small \( Bi \leq 0.1 \)). In this extreme scenario, most likely the 1-D composite lumped model does not need to be used because it collapses into the basic lumped model. Consistent with the criterion for the basic lumped model, we selected a combination of \( Bi = 0.1 \) and \( \tau = 1.4 \). For this limiting condition, the mean, internal Nusselt number from equation (A.9) is \( \bar{Nu}_s = 7.07 \) and the mean overall Nusselt number from equation (14) is \( \bar{Nu}_{ov} = 0.19 \). After evaluating equations (17), (18) and (28), the calculated trio of dimensionless temperatures corresponds to: \( \theta_m = 0.66 \), \( \theta_w = 0.65 \) and \( \theta_c = 0.69 \). In contrast, the evaluation of equation (5) brings forth \( \theta_m = 0.66 \) and \( \theta_c = 0.68 \), respectively. The two local temperatures \( \theta_w \) and \( \theta_c \) are perfectly harmonious exhibiting a negligible deviation of 0.01 units only.

8. Conclusions

Based on the one-to-one comparisons for the relevant mean, surface and center temperatures, \( T_m, T_w \) and \( T_c \) in the demanding sphere using the new 1-D composite lumped model for three distinct heat exchange intensities (vigorous, moderate and weak), it may be attested without reservations that the three dimensionless temperatures \( T_m, T_w \) and \( T_c \) can be determined with elementary algebraic equations for all time \((0 < t < \infty)\). This demonstrates that evaluation of the Fourier infinite series representative of the spatio-temporal temperature is unnecessary. The new methodology should work well for the other two simple bodies (the large slab and the long cylinder), and must exhibit equal or superior accuracy.

References


Appendix: The Two Fundamental Sub-Problems

Fundamental sub-problem 1: Unsteady 1-D heat conduction in a solid sphere with prescribed surface temperature, $T_w$.

Adopting the customary dimensionless temperature variable

$$\psi = \frac{T - T_w}{T_{in} - T_w}$$  \hspace{1cm} (A.1)

along with the dimensionless variable $\eta$ and the dimensionless time $\tau$ from equation (4), the 1-D heat conduction equation may be written in dimensionless form as

$$\frac{\partial \psi}{\partial \tau} = \frac{\partial^2 \psi}{\partial \eta^2} + \frac{2}{\eta} \frac{\partial \psi}{\partial \eta}$$  \hspace{1cm} (A.2)

The initial condition is

$$\psi(\eta, 0) = 1$$  \hspace{1cm} (A.3)

and the boundary conditions are

$$\frac{\partial \psi(0, \tau)}{\partial \eta} = 0$$  \hspace{1cm} (A.4a)

$$\psi(1, \tau) = 0$$  \hspace{1cm} (A.4b)

The exact, analytic 1-D dimensionless temperature distribution, $\psi(\eta, \tau)$, as found in [2-8] is

$$\psi(\eta, \tau) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \left[ \frac{\sin(\mu_n \eta)}{\mu_n \eta} \right] \exp(-\mu_n^2 \tau)$$  \hspace{1cm} (A.5)
where the eigenvalues are $\mu_n = n\pi, (n = 1, 2, 3, \ldots)$. The companion dimensionless mean temperature $\psi_m(\tau)$ is computed with

$$\psi_m(\tau) = 3 \int_0^1 \psi(\eta,\tau) \eta^2 \, d\eta$$

using the mean value theorem.

With this information, the local, “internal Nusselt number”, $Nu_s$, being the dimensionless representation of the local, “internal convective coefficient”, $h_s$, is determined from the ratio between the dimensionless temperature derivative at the surface and the dimensionless mean temperature [6]. In equation form, this entails to

$$Nu_s(\tau) = \frac{h_s D}{k_s} = - \frac{2 \varphi_{(1,\tau)}}{\varphi_m(\tau)}$$

(A.7)

Fig. 2. Variation of the mean, “internal Nusselt number” $\overline{Nu_s}$ with the dimensionless time $\tau$ in a sphere with prescribed surface temperature

From here, the mean, “internal Nusselt number”, $\overline{Nu_s}$, is determined from

$$\overline{Nu_s} (\tau) = \frac{1}{\tau} \int_0^\tau Nu_s (\tau) \, d\tau$$

(A.8)

Recall that $\overline{Nu_s}$ is one of the two ingredient for the mean, overall Nusselt number, $\overline{Nu_{ov}}$, as seen in equation (11).
Applying regression analysis to the $\overline{Nu}_s$ versus $\tau$ data with the Minitab code [12], generates the compact correlation equation:

$$\overline{Nu}_s = \sqrt{46.255 + \frac{5.139}{\tau}}$$  \hspace{1cm} (A.9)

with a high correlation coefficients, $R^2 = 0.9998$. To facilitate the visualization, $\overline{Nu}_s$ versus $\tau$ is plotted in Figure 2.

**Fundamental sub-problem 2:** Steady heat convection from a solid sphere with uniform temperature to a surrounding cold fluid.

As we know, the other key ingredient in equation (11) is the mean, external convective coefficient, $\overline{h}_f$, which is normally channeled through the Biot number, $Bi$, for heat conduction analysis. Traditionally, numerical values of $\overline{Nu}_f$ have been determined from theoretical, numerical and experimental procedures for: forced convection and natural convection around bodies.

1) For forced convection over a solid sphere with uniform temperature, $T_w$, Whitaker [13] recommended the correlation equation

$$\overline{Nu}_f = 2 + \left( 0.4 \frac{Re^{3/2}}{Pr} + 0.06 \frac{Re^{3/2}}{Pr} \right) Pr^{0.4}$$  \hspace{1cm} (A.10)

for $3.5 < Re < 10^5$ and $Pr > 0.7$. All thermo-physical properties are evaluated at the free-stream temperature $T_f$.

2) For natural convection over a solid sphere with uniform temperature, $T_w$, Churchill [14] recommended the correlation equation

$$\overline{Nu}_f = 2 + \frac{0.59 Ra^{1/4}}{f(Pr)}$$  \hspace{1cm} (A.11)

for $0 < Ra < 10^{11}$ and $Pr > 0.5$, where the Prandtl number function is

$$f(Pr) = \left[ 1 + \left( \frac{0.47}{Pr} \right)^{9/16} \right]^{4/9}$$  \hspace{1cm} (A.11a)

All thermo-physical properties in equations (A.11) and (A.12) are evaluated at the film temperature $\frac{T_w + T_f}{2}$.