About Statistical Simulation of 4D Random Fields by Means of Kotelnikov-Shannon Decomposition

Zoya Vyzhva1* and Kateryna Fedorenko1

Received: 28 April 2016; Published online: 30 July 2016

© The author(s) 2016. Published with open access at www.uscip.us

Abstract

We consider the problem of statistical simulation of the mean square estimates for the real valued 4D random fields that are homogeneous with respect to the time parameter and homogeneous and isotropic with respect to the spatial variables. An analogue of the Kotelnikov-Shannon theorem for the random fields with a bounded spectrum is presented. Models of such random fields are constructed with the help of partial sums of some series. Some important estimates for the mean square approximation of a random field by its models are obtained. Statistical simulation procedures of realizations of a random field with Gaussian distribution are constructed. Using these theorems, models and procedures, we demonstrate applications of generating by means of computer adequate realizations of Gaussian random fields with examples of covariance functions.

Keywords: Gaussian Random Fields; Statistical Simulation; Kotelnikov-Shannon Decomposition; Procedure; Approximation; Mean Square Estimates

2010 Mathematics Subject Classification: Primary: 60G60, Secondary: 65C05, 65C20, 60G15

1. Introduction

Due to the rapid development of computer technology, methods of numerical simulation (the so called Monte Carlo methods) of stochastic processes and random fields have an expanding range of applications. They are applied, in particular, in such natural sciences as geology, geophysics, geoinformatics, seismology, meteorology, oceanography, electrical engineering, statistical radio physics, nuclear physics and others. Using statistical simulation techniques and computers, one can generate realizations of stochastic processes and random fields for which necessary statistical data is known.

The statistical simulation of random functions on the basis of their spectral decomposition is

*Corresponding e-mail: zoya_vyzhva@ukr.net
1 Department of General Mathematics, Faculty for Mathematics and Mechanics, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
developed by Yadrenko (see book by Yadrenko (1983)) and was investigated by several authors (see, for example, surveys in Yadrenko and Gamaliy, 1998; Vyzhva, 2003; 2011-2013).

The real valued random fields that are homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables in three-dimensional space are natural generalizations of random fields with bounded spectrum which are homogeneous with respect to time and homogeneous isotropic with respect to \( \varphi \) on the unit cylinder in \( \mathbb{R} \times S_2 \), random fields homogeneous with respect to time in \( \mathbb{R} \times S_3 \), where \( S_3 \) is an unit sphere in 3D Euclidean space and random fields homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables in 2D and 3D space.

Random fields which are homogeneous with respect to time on cylinder are considered in papers by Roch (1972), Moklyachuk (1974) and random fields which are homogeneous with respect to time in \( \mathbb{R} \times S_n \), where \( S_n \) is the unit sphere in \( n \)-dimensional Euclidean space \( (n \geq 2) \) are considered in papers by Moklyachuk and Yadrenko (1978).

Modified Kotelnikov-Shannon interpolation decompositions of stochastic processes and multivariate random fields was studied by many authors (Belyayev, 1959; Piranashvili, 1967; Higgins, 1996; Seip, 2004; Butzer, 2012; Vyzhva, 2003, 2011-2013; Olenko, 2004, 2005, 2011; Pogany, 2005, 2011; Vyzhva and Fedorenko, 2014). An analogue of the Kotelnikov-Shannon theorem was applied to random fields with a bounded spectrum that are time homogeneous and isotropic on cylinder in \( \mathbb{R} \times S_2 \) in paper by Halikulov and Yadrenko (2000) and to random fields with a bounded spectrum that are time homogeneous and isotropic in a unit sphere in \( n \)-dimensional Euclidean space \( (n \geq 2) \) in paper by Halikulov and Vyzhva (2001). The statistical simulation of 3D random fields by means of Kotelnikov-Shannon decomposition is considered in paper by Vyzhva and Fedorenko (2014), where the problem of the mean square estimates of the real valued random fields that are homogeneous with respect to time and homogeneous isotropic with respect to the 2D spatial variables is resolved.

This paper deals with real valued and mean square continuous 4D random fields \( \xi(t, x) = \xi(t, r, \theta, \varphi) \) in \( \mathbb{R} \times \mathbb{R}^3 \) which are homogeneous with respect to the time variable \( t \) and homogeneous isotropic with respect to the variables \( r, \theta, \varphi \). The spectral representation theorem for such random fields is proved. An analogue of the Kotelnikov-Shannon theorem for random fields whose spectrum is bounded in time is presented. The model and simulating procedure for Gaussian random fields with given statistical characteristics are constructed.

2. Homogeneous with Respect to Time and Homogeneous Isotropic with Respect to Spatial Variables 4D Random Fields

Consider a real valued mean square continuous 4D random field \( \xi(t, x) = \xi(t, r, \theta, \varphi), t \in \mathbb{R}, x \in \mathbb{R}^3 \).
Definition 2.1. (homogeneous with respect to time and homogeneous isotropic with respect to spatial variables random field) The 4D random field $\xi(t, r, \theta, \varphi)$, $t \in R$, $x \in R^3$ is called homogeneous with respect to time and homogeneous isotropic with respect to spatial variables $r, \theta, \varphi$ (here $r, \theta, \varphi$ are polar coordinates of a point $x$ and $r \in R$, $\theta \in [0, \pi], \varphi \in [0, 2\pi]$), if it satisfies the following conditions:

1) $E\xi(t, x_1) = \text{const}$ for all $t \in R$ and $x_1 \in R^3$ (we assume that $E\xi(t, x_1) = 0$).

2) $E\xi(t, x_1)\xi(s, x_2) = B(t-s, \rho)$ for all $t, s \in R$ and $x_1, x_2 \in R^3$,

where $B(t, \rho)$ is the correlation function that depends on the shift of the time $t = t-s$ and the distance between vectors $x_1$ and $x_2$, that is $\rho = |x_1 - x_2|$. The distance between the point $x_1 = (r_1, \theta_1, \varphi_1)$ and the point $x_2 = (r_2, \theta_2, \varphi_2)$ equals $\rho = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\psi}$, where $\cos\psi$ the angular distance between points $x_1$ and $x_2$: $\cos\psi = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2)$.

The spectral representation of the correlation function of a real valued random field $\xi(t, x)$ in $R \times R^n$ which is homogeneous with respect to the time $t$ and homogeneous isotropic with respect to the spatial variables is defined (Yadrenko, 1983) as integral

$$B(t-s, \rho) = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} e^{i (t-s)u} Y_n(\lambda \rho) \Phi(du, d\lambda),$$  

(1)

where $Y_n(\lambda \rho) = 2^{-\frac{n-2}{2}} \frac{n}{2} J_{n-2}(\lambda \rho)(\lambda \rho)^{-\frac{n-2}{2}}$, and $\Phi(du, d\lambda)$ is the spectral measure of the random field, and $J_m(x)$ is the Bessel function of the first kind of order $m$.

The correlation function (1) for dimension $n = 3$ is represented by the formula

$$B(t-s, \rho) = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} e^{i (t-s)u} \frac{1}{2\pi} \frac{1}{2} \Phi(du, d\lambda).$$  

(2)

It is easy to check that $J_1(t) = \sqrt{\frac{2}{\pi}} \frac{\sin t}{\sqrt{t}}$ and the next expression holds in this case

$$B(t-s, \rho) = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} e^{i (t-s)u} \frac{\sin(\lambda \rho)}{\lambda \rho} \Phi(du, d\lambda).$$  

(3)

The following spectral representation theorem holds true.
Theorem 2.1. A mean square continuous 4D random field $\xi(t, r, \theta, \phi)$ in $R \times R^3$ which is time homogeneous and homogeneous isotropic with respect to the spatial variables admits the following spectral decomposition

$$
\xi(t, r, \theta, \phi) = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} c_{m,l} P^l_m(\cos \theta) \left[ \zeta^l_{m,1}(t, r) \cos l\phi + \zeta^l_{m,2}(t, r) \sin l\phi \right],
$$

(4)

where sequences of 2D random fields are of the form

$$
\zeta^l_{m,k}(t, r) = \int_{-\infty}^{\infty} \int_{0}^{\infty} J_{m+\frac{1}{2}}(\lambda r) e^{imx} \frac{Z^l_{m,k}(du, d\lambda)}{\sqrt{\lambda r}}, \quad m = 0, 1, \ldots; l = 0, 1, \ldots, m; \quad k = 1, 2,
$$

(5)

constants are calculated by the formula

$$
c_{m,l} = \frac{\pi \nu_l (2m+1)(m-l)!}{2 (m+l)!}, \quad \nu_l = \begin{cases} 1, & l = 0 \\ 2, & l > 0, \end{cases}
$$

and $P^l_m(x)$ are associated Legendre functions of degree $m$, and $\{Z^l_{m,1}(\cdot), Z^l_{m,2}(\cdot)\}$ are sequences of real valued orthogonal random measure on Borel subsets of the set $(-\infty, +\infty) \times [0, +\infty)$ which satisfy the next conditions:

$$
EZ_{t,k}^l(S_1) = 0, \quad EZ_{t,k}^l(S_1)Z^q_{p,j}(S_2) = \delta^e_j \delta^p_l \delta^q \Phi(S_1 \cap S_2),
$$

(6)

for all Borel subsets $S_1$ and $S_2$ of $R \times R, t, p = 0, 1, \ldots; l, q = 0, 1, \ldots, m; \quad k, j = 1, 2$.

Proof: If $\rho = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos \psi}$, then by using the addition theorem for Bessel functions we have the next expression

$$
J_{\nu}(\rho) = \left( \frac{r_1r_2}{2} \right)^{-\nu} \Gamma(\nu) \sum_{m=0}^{\infty} (\nu + m) C^\nu_m(\cos \psi) J_{\nu+m}(r_1) J_{\nu+m}(r_2),
$$

(7)

where $C^\nu_m(z)$ are Gegenbauer polynomials (see Erdélyi et al (1953)) which are defined in terms of their generating function
Then by the addition theorem for Bessel functions (7) the correlation function of random field \( \xi(t, r, \theta, \varphi) \) (2) admits the next decomposition

\[
E \xi(t, x_1) \xi(s, x_2) = \pi \sum_{m=0}^{\infty} \left( m + \frac{1}{2} \right) C_m^\frac{1}{2} (\cos \psi) \int_{-\infty}^{+\infty} e^{i(t-s)u} J_{\frac{1}{2}+m} (\lambda r_1) J_{\frac{1}{2}+m} (\lambda r_2) \frac{\sqrt{\lambda r_1}}{\sqrt{\lambda r_2}} \Phi(du, d\lambda). \tag{9}
\]

Note, that \( C_m^\frac{1}{2} (\cos \psi) = P_m (\cos \psi), \) \( C_m^\nu (1) = \frac{\Gamma(m+2\nu)}{m!\Gamma(2\nu)}, \) \( P_m(\cdot) \) are associated Legendre functions of degree \( m \). Then we use the addition theorem for Legendre polynomials and have

\[
C_m^\frac{1}{2} (\cos \psi) = P_m (\cos \psi) = P_m \left( \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos (\varphi_1 - \varphi_2) \right) =
\]

\[
= P_m (\cos \theta_1) P_m (\cos \theta_2) + 2 \sum_{l=1}^{m} \frac{(m-l)!}{(m+1)!} P_m^l (\cos \theta_1) P_m^l (\cos \theta_2) \cos l (\varphi_1 - \varphi_2) =
\]

\[
= \sum_{l=0}^{m} \nu_l \frac{(m-l)!}{(m+1)!} P_m^l (\cos \theta_1) P_m^l (\cos \theta_2) \cos l (\varphi_1 - \varphi_2).
\]

Applying the previous result and formula (9) we obtain the next expression for correlation function

\[
E \xi(t, x_1) \xi(s, x_2) = \sum_{m=0}^{\infty} \sum_{l=0}^{m} \pi \left( m + \frac{1}{2} \right) \nu_l \frac{(m-l)!}{(m+1)!} P_m^l (\cos \theta_1) P_m^l (\cos \theta_2) \times
\]

\[
\cos l (\varphi_1 - \varphi_2) \int_{-\infty}^{+\infty} e^{i(t-s)u} J_{\frac{1}{2}+m} (\lambda r_1) J_{\frac{1}{2}+m} (\lambda r_2) \frac{\sqrt{\lambda r_1}}{\sqrt{\lambda r_2}} \Phi(du, d\lambda).
\]

Now we apply the Karhunen theorem and prove the decomposition (4) for the random field \( \xi(t, r, \theta, \varphi) \).

Since \( E \xi(t, r, \theta, \varphi) = 0 \) we have \( E \xi_{m,k}^l (t, r) = 0, m = 0, 1, \ldots; l = 0, 1, \ldots, m; k = 1, 2. \)

If one considers the restriction of the random field \( \xi(t, r, \theta, \varphi) \) to the sphere of a fixed radius \( r \) and applies the theorem about properties of spherical harmonics (see book by Yadrenko (1983)) to relation (9), then he finds that the correlation function of the random field is of the form

\[
(1-2zy+y^2)^{-\nu} = \sum_{m=0}^{\infty} C_m^\nu (z) y^m, |y| < |z + \sqrt{z^2 - 1}|.
\]
where $S^l_m(\theta, \varphi)$ are orthonormal spherical harmonics of degree $m, l = 1, \ldots, h(m, n)$.

The coefficients

$$b_m(t - s, r) = \int_{-\infty}^{\infty} \int_{0}^{+\infty} e^{i(t-s)u} \frac{J^2_{1+m}(\lambda r)}{\lambda r} \Phi(du, d\lambda),$$

of decomposition (10) are called the spectral coefficients.

**Theorem 2.2.** If $\xi(t, r, \theta, \varphi)$ is a random field in $R \times R^3$ which is homogeneous in time and homogeneous isotropic with respect to the spatial variables, then

$$E\xi_{m, p}^l(t, r)\xi^k_{q, j}(s, r) = \delta^q_m \delta^k_p b_m(t - s, r),$$

where $\delta^r_k$ is the Kronecker symbol, $\{b_m(t - s, r), m = 0, 1, 2, \ldots\}$ is a sequence of positive definite kernels in $R \times R_+$ of the form (11) and such that

$$\sum_{m=0}^{\infty} \left(m + \frac{1}{2}\right) b_m(t - s, r) < \infty.$$

The variance of the random field $\xi(t, r, \theta, \varphi)$ is given as

$$\text{Var} \xi(t, r, \theta, \varphi) = \pi \sum_{m=0}^{\infty} \left(m + \frac{1}{2}\right) b_m(0, r).$$

**Proof.** Properties (12) follows from condition (6) and formula for spectral coefficients (11).

The variance of random field $\xi(t, r, \theta, \varphi)$ is calculated by formula (9)

$$E\xi(t, x)\xi(t, x) = \pi \sum_{m=0}^{\infty} \left(m + \frac{1}{2}\right) C^2_m(1) b_m(0, r) = \pi \sum_{m=0}^{\infty} \left(m + \frac{1}{2}\right) b_m(0, r).$$

Theorem is proved.
The expansion (4) can be used for statistical simulation of 4D random fields \( \xi(t, r, \theta, \varphi) \) in \( R \times R^3 \) which are homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables if the spectral function (or correlation function) is specified.

5.1 Time homogeneous random field with a bounded in time spectrum

We consider a random field \( \xi(t, r, \theta, \varphi) \) in \( R \times R^3 \).

**Definition 2.2. (random field with a bounded spectrum)** We say that \( \xi(t, r, \theta, \varphi) \) is a random field with a bounded spectrum with respect to time \( t \) if all its spectral measures are concentrated on \((-\omega, \omega] \times R_\circ \).

The following result is due to Belyaev (1959).

**Lemma 2.3.** If \( \xi(t) \) is a stationary process with the spectral function which is concentrated on the interval \([-\omega, \omega] \), \( \omega > \tilde{\omega} \) and if

\[
\xi_N(t) = \sum_{k=-N}^N \xi \left( \frac{k \pi}{\omega} \right) \frac{\sin \left( t \frac{k \pi}{\omega} \right)}{\frac{k \pi}{\omega}} ,
\]

then

\[
E \left| \xi(t) - \xi_N(t) \right|^2 \leq \frac{\gamma^2(t)}{N^2} \frac{\sigma^2}{1 - \frac{\omega}{\tilde{\omega}}} ,
\]

where \( \gamma(t) = \frac{4}{\pi} \frac{\omega}{|t| + 1} \), \( \sigma^2 = \text{Var} \xi(t) \).

Let \( \xi(t, r, \theta, \varphi), t \in R, r \in R_\circ, \theta \in [0, \pi], \varphi \in [0, 2\pi] \) be a 4D random field on \( R \times R^3 \) which is time homogeneous and homogeneous isotropic with respect to the spatial variables. Assume that the spectrum \( \Phi(U, \Lambda) \) of the field \( \xi(t, r, \theta, \varphi) \) is bounded with respect to time \( t \), \( U \subset \left[ -\omega, \omega \right], \Lambda \subset R_\circ \); and let \( \Phi(U, \Lambda) \) be concentrated on \([-\omega, \omega] \times R_\circ \).

Let \( \omega \) be an arbitrary number such that \( \omega > \omega \). We consider an analogue of partial sum (14) for the random field and represent it in the form
The following assertion holds.

**Theorem 2.4.** Let \( \xi(t,r,\theta,\varphi) \) be a 4D random field in \( \mathbb{R} \times \mathbb{R}^3 \) which is time homogeneous and homogeneous isotropic with respect to the spatial variables. If the spectrum of \( \xi(t,r,\theta,\varphi) \) is bounded in time \( t \), then the mean square approximation by the partial sum (16) is such that

\[
E \left[ |\xi(t,r,\theta,\varphi) - \xi_N(t,r,\theta,\varphi)|^2 \right] \leq \frac{\gamma^2(t)}{N^2} \left( \frac{1}{1 - \frac{\omega}{\omega}} \right)^2 \left( \pi \sum_{m=0}^{\infty} \left( m + \frac{1}{2} \right) \tilde{b}_m(0,r) \right),
\]

where

\[
\tilde{b}_m(0,r) = \int_{-\omega}^{\omega} \int_0^{\infty} \frac{J^2_{\frac{1}{2}+m}(\lambda r)}{\lambda r} \Phi(du,d\lambda).
\]

**Proof.** Indeed, we use Theorem 2.2 and boundedness of the spectrum. Then Belyaev's Lemma implies inequality (17). Theorem is proved.

**Corollary 2.5.** Let \( \xi(t,r,\theta,\varphi) \) be a 4D random field in \( \mathbb{R} \times \mathbb{R}^3 \) with bounded spectrum in time \( t \). Then \( \xi(t,r,\theta,\varphi) \) admits the following Kotelnikov-Shannon decomposition

\[
\xi(t,r,\theta,\varphi) = \sum_{k=-\infty}^{\infty} \xi \left( \frac{k\pi}{\omega}, r, \theta, \varphi \right) \sin \left( \frac{t - \frac{k\pi}{\omega}}{\omega} \right),
\]

where the series on the right hand side of (19) converges in the mean square sense for \( \omega > \omega \).

The approximation theorem for 4D random field in \( \mathbb{R} \times \mathbb{R}^3 \) which is time homogeneous and homogeneous isotropic with respect to the spatial variables and random fields with bounded in time spectrum is proved similarly as in work Vyzhva and Fedorenko (2014).
3. Estimates of the Mean Square Approximation of Time Homogeneous and Homogeneous Isotropic with Respect to the Spatial Variables 4D Random Fields

Let us give some properties of spectral coefficients (11) by means of results from the book by Watson (1945) and prove the next lemma.

**Lemma 3.1.** If \( \tilde{b}_m(0, \rho) \) is evaluated by formula (18) then the following properties hold true

\[
\sum_{m=0}^{\infty} (2m+1) \tilde{b}_m(0, r) = \frac{2}{\pi} \mu_0, \tag{20}
\]

\[
\sum_{m=0}^{\infty} (2m+1) \tilde{b}_m(0, r) \leq \frac{2^{s+1} r^s (s!)^2}{(2s)! (2s+1)! \pi} \mu_{2s}, \tag{21}
\]

where

\[
\mu_k = \int_{-\omega}^{+\omega} \int_0^{+\infty} \lambda^k \Phi(du, d\lambda).
\]

**Proof.** Let us use the following statements from the book by Watson (1945)

\[
J_m^2(z) = \frac{2}{\pi} \int_0^\pi J_{2m}(2z \sin \theta)d\theta, \tag{22}
\]

\[
\left( \frac{z}{2} \right)^{2s+1} = \sum_{m=s}^{\infty} \frac{(2m+1)(s+m)!}{(m-s)!} J_{2m+1}(z), s = 0,1,\ldots, \tag{23}
\]

\[
\left( \frac{z}{2} \right)^s = \sum_{m=0}^{\infty} \frac{(s+2m)(s+m-1)!}{m!} J_{s+2m}(z), s = 1,2,\ldots, \tag{24}
\]

Applying relations (22) and (23) we derive equality (20) as

\[
\sum_{m=0}^{\infty} (2m+1) \tilde{b}_m(0, r) = \int_{[\rho]}^{+\infty} \int_0^{+\infty} \left( \frac{\lambda r}{2 + m} \right)^{-1} \sum_{m=0}^{\infty} (2m+1) J_m^2(\lambda r) \Phi(du, d\lambda) =
\]
Then we derive inequality (21) by using formula (22) as follows
\[
\sum_{m=s}^{\infty}(2m+1)\tilde{b}_m(0, r) = \int_{|\lambda|<\infty}^{+\infty}(\lambda r)^{-1} \int_{0}^{\pi} \int_{0}^{\pi} (2\lambda r \sin \theta) d\theta d\lambda = \\
\int_{|\lambda|<\infty}^{+\infty}(\lambda r)^{-1} \int_{0}^{\pi} \int_{0}^{\pi} (2\lambda r \sin \theta) d\theta d\lambda.
\]

Then we use \((m+2s)! \geq n!\), \( \forall m, s \geq 0\), the formula (24) and the following inequality holds
\[
\sum_{m=s}^{\infty}(2m+1)J_{2m+1}(2\lambda r \sin \theta) \leq \\
\frac{1}{(2s)!} \sum_{m=0}^{\infty}(2m+2s+1) J_{2m+2s+1} (2\lambda r \sin \theta) = \\
\frac{(\lambda r \sin \theta)^{2s+1}}{(2s)!}.
\]

We use the derived formulas and formulas 3.621 (1) and 8.384 (1) from the book by Gradshteyn et al (2000) and obtain the inequality (21) as
\[
\sum_{m=s}^{\infty}(2m+1)\tilde{b}_m(0, r) \leq \\
\frac{2}{\pi} \int_{|\lambda|<\infty}^{+\infty}(\lambda r)^{-1} \int_{0}^{\pi} \int_{0}^{\pi} (\lambda r \sin \theta)^{2s+1} d\theta d\lambda = \\
\frac{2^{2s}}{(2s)!} \pi \int_{|\lambda|<\infty}^{+\infty} \lambda^{-1} \int_{0}^{\pi} \int_{0}^{\pi} (\sin \theta)^{2s+2} d\theta d\lambda = \\
\frac{2^{2s+2} B(s+1, s+1) \mu_{2s}}{(2s)! \pi} = \frac{2^{2s+1} r^{2s} \Gamma^2(s+1)}{(2s)! \pi} \Gamma(2s+2) \mu_{2s} = \\
\frac{2^{2s+1} r^{2s} (s!)^2}{(2s)! (2s+1)! \pi} \mu_{2s}.
\]

Thus the statements of the lemma hold true.
We use the partial sum (4) and the partial sum of the decomposition (16) for a random field \( \xi(t, r, \theta, \varphi) \) to construct a model for such field if its spectrum is bounded with respect to time \( t \).

The approximating model is written as follows

\[
\xi_{N,M}(t, r, \theta, \varphi) = \sum_{m=0}^{M} \sum_{l=0}^{L} c_{m,l} P_{m}^{l}(\cos \theta) \times \\
\left[ \cos \varphi \sum_{k=-N}^{N} \frac{\sin \omega \left( t - \frac{k \pi}{\omega} \right)}{\omega \left( t - \frac{k \pi}{\omega} \right)} \xi_{m,1}^{l} \left( \frac{k \pi}{\omega}, r \right) + \sin \varphi \sum_{k=-N}^{N} \frac{\sin \omega \left( t - \frac{k \pi}{\omega} \right)}{\omega \left( t - \frac{k \pi}{\omega} \right)} \xi_{m,2}^{l} \left( \frac{k \pi}{\omega}, r \right) \right],
\tag{26}
\]

where \( \xi_{m,j}^{l} \left( \frac{k \pi}{\omega}, r \right) \) are values of the Gaussian stochastic processes for all \( m, p = 0,1, \ldots, M; k, q = -N, N; l, s = 0,1, \ldots, m; i, j = 1,2 \) such that:

\[
\xi_{m,j}^{l} \left( \frac{k \pi}{\omega}, r \right) = 0, \ E_{m,j}^{l} \left( \frac{k \pi}{\omega}, r \right) \xi_{p,i}^{s} \left( \frac{q \pi}{\omega}, r \right) = \delta_{i}^{j} \delta_{p}^{m} \xi_{m,j}^{l} \left( \frac{(k-q)\pi}{\omega}, r \right),
\]

where \( \tilde{b}_{m}(t-s, r) \) is evaluated by formula (18).

**Theorem 3.2.** Let \( \xi(t, r, \theta, \varphi) \) be a 4D random field in \( R \times R^{3} \) which is homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables. If the spectrum of \( \xi(t, r, \theta, \varphi) \) is bounded in time \( t \) and all its spectral measures are concentrated in \( \left[ -\omega, \omega \right] \times R_{+} \), then the mean square approximation by model (26) is such that

\[
E \left[ \xi(t, x) - \xi_{N,M}(t, x) \right]^{2} \leq \frac{5 \pi r^{3}}{2 M^{2}} \mu_{3} + \frac{\gamma^{2}(t)}{N^{2}} \left( 1 - \frac{\omega}{\omega} \right)^{2} \left( 2 \mu_{0} \frac{5 \pi r^{3}}{2 M^{2}} \mu_{3} \right) < \varepsilon, \tag{27}
\]

and

\[
E \left[ \xi(t, x) - \xi_{N,M}(t, x) \right]^{2} \leq 
\]
where \( r, \theta, \varphi \) are polar coordinates of a point \( x \), \( \omega \) is an arbitrary number such that \( \omega > \omega' \), function \( \gamma(t) = \frac{4}{\pi} \left( \frac{\omega}{\pi} |t| + 1 \right) \) and

\[
\mu_k = \int_{-\omega}^{+\omega} \int_0^{\infty} \lambda^k \Phi(du, d\lambda). \tag{29}
\]

**Proof.** Recall that a random field \( \xi(t, r, \theta, \varphi) \) in \( \mathbb{R} \times \mathbb{R}^3 \) which is time homogeneous and homogeneous isotropic with respect to the spatial variables admits the decomposition (4) into two sums, that is

\[
\xi(t, r, \theta, \varphi) = \sum_{m=0}^{\infty} \sum_{l=0}^{m} c_{m,l} P_m^l(\cos \theta) \left[ \zeta_{m,1}^l(t, r) \cos \varphi + \zeta_{m,2}^l(t, r) \sin \varphi \right] +
\]

\[
\sum_{m=M+1}^{\infty} \sum_{l=0}^{m} c_{m,l} P_m^l(\cos \theta) \left[ \zeta_{m,1}^l(t, r) \cos \varphi + \zeta_{m,2}^l(t, r) \sin \varphi \right]. \tag{30}
\]

Then we consider the difference between the random field \( \tilde{\xi}(t, r, \theta, \varphi) \) and the model (26) as

\[
\tilde{\xi}(t, r, \theta, \varphi) = \sum_{m=M+1}^{\infty} \sum_{l=0}^{m} c_{m,l} P_m^l(\cos \theta) \left[ \zeta_{m,1}^l(t, r) \cos \varphi + \zeta_{m,2}^l(t, r) \sin \varphi \right] +
\]

\[
\sum_{m=0}^{M} \sum_{l=0}^{m} c_{m,l} P_m^l(\cos \theta) \left[ \zeta_{m,1}^l(t, r) \cos \varphi - \sum_{k=0}^{N} \sin \omega \left( t \frac{k \pi}{\omega} \right) \zeta_{m,1}^l \left( \omega \frac{k \pi}{\omega}, r \right) \right]
\]
\[
\sin l\varphi \left\{ \zeta_{m,2}^l (t, r) - \sum_{k=-N}^{N} \frac{\sin \omega \left( t - \frac{k\pi}{\omega} \right)}{\omega \left( t - \frac{k\pi}{\omega} \right)} \zeta_{m,2}^l \left( \frac{k\pi}{\omega}, r \right) \right\}.
\]

Passing to the mathematical expectations of the square of the last expression and taking into account the mutual orthogonality of stochastic processes \( \zeta_{m,1}^l (t, r) \) and \( \zeta_{m,2}^l (t, r) \), inequality \((a+b)^2 \leq 2a^2 + 2b^2\) and Belyaev’s Lemma we obtain the mean square approximation of random field \( \xi(t,r,\theta,\varphi) \) by model (26)

\[
E \left| \xi(t,r,\theta,\varphi) - \xi_{N,M}(t,r,\theta,\varphi) \right|^2 \leq 2\pi \sum_{m=M+1}^{\infty} \left( m + \frac{1}{2} \right) \tilde{b}_m (0, r) + \\
2 \sum_{m=0}^{M} \sum_{l=0}^{m} \left( c_{m,l} P_l (\cos \theta) \right)^2 \left[ \cos 2l\varphi E \left| \zeta_{m,1}^l (t, r) - \sum_{k=-N}^{N} \frac{\sin \omega \left( t - \frac{k\pi}{\omega} \right)}{\omega \left( t - \frac{k\pi}{\omega} \right)} \zeta_{m,1}^l \left( \frac{k\pi}{\omega}, r \right) \right|^2 + \\
\sin 2l\varphi E \left| \zeta_{m,2}^l (t, r) - \sum_{k=-N}^{N} \frac{\sin \omega \left( t - \frac{k\pi}{\omega} \right)}{\omega \left( t - \frac{k\pi}{\omega} \right)} \zeta_{m,2}^l \left( \frac{k\pi}{\omega}, r \right) \right|^2 \right] = \\
2\pi \sum_{m=M+1}^{\infty} \left( m + \frac{1}{2} \right) \tilde{b}_m (0, r) + \frac{\gamma^2(t)}{N^2} \frac{\pi}{2} \sum_{m=0}^{M} \left( m + \frac{1}{2} \right) \tilde{b}_m (0, r) = \\
\pi \sum_{m=M+1}^{\infty} \left( 2m+1 \right) \tilde{b}_m (0, r) + \pi \frac{\gamma^2(t)}{N^2} \frac{1}{1 - \frac{\omega}{\omega}} \left( \sum_{m=0}^{\infty} \left( 2m+1 \right) \tilde{b}_m (0, r) - \sum_{m=M+1}^{\infty} \left( 2m+1 \right) \tilde{b}_m (0, r) \right). (31)
\]

We estimate the first term in the last inequality by using the Yadrenko and Gamaliy (1998) Lemma stated below.
Lemma 3.3. We have the next inequalities for the Bessel functions of the first kind

\[ \sum_{m=M+1}^{\infty} J_{\theta+m}^2(z) \leq \frac{z^2}{M}, \quad \theta \in [0,1), \]

and

\[ \sum_{m=M+1}^{\infty} \left( m + \frac{1}{2} \right) J_{\frac{1}{2}+m}^2(z) \leq \frac{5z^4}{4M^2}. \]

Applying the result of previous lemma we obtain the next evaluation

\[ 2\pi \sum_{m=M+1}^{\infty} \left( m + \frac{1}{2} \right) \hat{b}_m(0,r) = 2\pi \int_{-\infty}^{+\infty} \sum_{m=M+1}^{\infty} \left( m + \frac{1}{2} \right) \frac{J_{\frac{1}{2}+m}(\lambda r)}{\lambda r} \Phi(du,d\lambda) \leq \]

\[ 2\pi \int_{-\infty}^{+\infty} \frac{5(\lambda r)^4}{4M^2\lambda r} \Phi(du,d\lambda) \leq \frac{5\pi r^3}{2M^2} \mu_3, \quad (32) \]

Finally, we use the inequalities (21), (32), the equality (20) and obtain the mean square estimates for the approximation of a random field \( \xi(t,r,\theta,\varphi) = \xi(t,x) \) in \( R \times R^3 \) which is time homogeneous and homogeneous isotropic with respect to the spatial variables by model (26) if the spectrum of \( \xi \) is bounded in \( t \). The estimates are written as follow

\[ E \left| \xi(t,x) - \xi_{N,M}(t,x) \right|^2 \leq \frac{5\pi r^3}{2M^2} \mu_3 + \frac{\gamma^2(t)}{N^2} \frac{1}{1 - \frac{\omega}{\omega_0}} \left( 2\mu_0 - \frac{5\pi r^3}{2M^2} \mu_3 \right) < \varepsilon, \]

and

\[ E \left| \xi(t,x) - \xi_{N,M}(t,x) \right|^2 \leq \frac{2^{2M+3}r^{2M+2}((M+1)!)^2}{(2M+2)!(2M+3)!} \frac{\mu_{2M+2} + \frac{\gamma^2(t)}{N^2}}{1 - \frac{\omega}{\omega_0}} \left( 2\mu_0 - \frac{2^{2M+3}r^{2M+2}((M+1)!)^2}{(2M+2)!(2M+3)!} \mu_{2M+2} \right) < \varepsilon. \]

The second term of these estimates has the order of convergence \( O\left( \frac{1}{N^2} \right) \). We can conclude that the statements of the theorem hold true.
3.1 Other estimates of the mean square approximation of 4D random fields of order $O\left(\frac{1}{N^2}\right)$

We use corollary (see Piranashvili (1967)) for a wide sense stationary stochastic process with the correlation function

$$B(t-s) = \int_{\Lambda} e^{i(t-s)u} F(du)$$

to estimate the mean square approximation of the random field $\xi(t, r, \theta, \varphi)$ by its model (26), where $\Lambda$ is a bounded domain of real numbers.

**Theorem 3.4.** Let $\xi(t, r, \theta, \varphi)$ be a 4D random field in $\mathbb{R} \times \mathbb{R}^3$ which is homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables. If the spectrum of $\xi(t, r, \theta, \varphi)$ is bounded in time $t$ and all its spectral measures are concentrated in $[-\omega, \omega] \times \mathbb{R}$, then the mean square approximation by model (26) is such that

$$E|\xi(t, x) - \xi_{N,M}(t, x)|^2 < \frac{5\pi r^3}{2M^2} \mu_3 + \frac{L_0^2(t)\omega^2}{(\omega - v)^2 N^2} \left(2\mu_0 - \frac{5\pi r^3}{2M^2} \mu_3\right) < \varepsilon, \quad (33)$$

and

$$E|\xi(t, x) - \xi_{N,M}(t, x)|^2 < \frac{2^{2M+3}r^{2M+2}((M+1)!)^2}{(2M+2)!(2M+3)!} \mu_{2M+2} + \frac{L_0^2(t)\omega^2}{(\omega - v)^2 N^2} \left(2\mu_0 - \frac{2^{2M+3}r^{2M+2}((M+1)!)^2}{(2M+2)!(2M+3)!} \mu_{2M+2}\right) < \varepsilon. \quad (34)$$

where $L_f = \sup_{u \in \Lambda, s \leq t} |e^{iu}| = 1$, $L_0(t) = \frac{2}{1-e^{-\pi}} \left(\frac{2}{\pi}\right) \sin \omega |t|$, $B(0) = \int_{\Lambda} F(du) = E|\xi(t)|^2$ and where $\omega > v = \sup_{u \in \Lambda} |u|$ is an arbitrary fixed number and $\mu_k$ is evaluated by formula (29).

**Proof.** For convenience we state below the result of corollary Piranashvili (1967)

$$E|\xi(t) - \xi_N(t)|^2 < \frac{L_j L_0^2(t)\omega^2}{(\omega - v)^2 N^2} B(0). \quad (35)$$
Using inequalities (31) and (35) we obtain the mean square estimate for the approximation of a random field \( \xi(t,r,\theta,\varphi) \) in \( \mathbb{R} \times \mathbb{R}^3 \) which is time homogeneous and homogeneous isotropic with respect to spatial variables by model (26) if the spectrum of \( \xi(t,r,\theta,\varphi) \) is bounded in \( t \). The estimate is written as follows

\[
E | \xi(t,r,\theta,\varphi) - \xi_{N,M}(t,r,\theta,\varphi) |^2 < \pi \sum_{m=M+1}^{\infty} (2m+1)\tilde{b}_m(0,r) + \\
\pi \frac{L_0^2(t)\omega^2}{(\omega - \nu)^2 N^2} \left( \sum_{m=0}^{\infty} (2m+1)\tilde{b}_m(0,r) - \sum_{m=M+1}^{\infty} (2m+1)\tilde{b}_m(0,r) \right).
\]

Finally, we use the inequalities (21), (32), the equality (20) and obtain the mean square estimates for the approximation of a random field \( \xi(t,r,\theta,\varphi) = \xi(t,x) \) in \( \mathbb{R} \times \mathbb{R}^3 \) which is time homogeneous and homogeneous isotropic with respect to the spatial variables by model (26) if the spectrum of \( \xi \) is bounded in \( t \). The finding estimates are written as follow

\[
E | \xi(t,x) - \xi_{N,M}(t,x) |^2 < \frac{5\pi r^3}{2M^2} \mu_3 + \frac{L_0^2(t)\omega^2}{(\omega - \nu)^2 N^2} \left( 2\mu_0 - \frac{5\pi r^3}{2M^2} \mu_3 \right) < \varepsilon,
\]

and

\[
E | \xi(t,x) - \xi_{N,M}(t,x) |^2 < \\
\frac{2^{2M+3}r^{2M+2}((M+1)!)^2}{(2M+2)!(2M+3)!} \mu_{2M+2} + \frac{L_0^2(t)\omega^2}{(\omega - \nu)^2 N^2} \left( 2\mu_0 - \frac{2^{2M+3}r^{2M+2}((M+1)!)^2}{(2M+2)!(2M+3)!} \mu_{2M+2} \right) < \varepsilon.
\]

We can conclude that the second term of these estimates has the order of convergence \( O \left( \frac{1}{N^2} \right) \).

The inequalities (33) and (34) hold true.

3.2 Estimates of the mean square approximation of 4D random fields of order \( O \left( \frac{1}{N} \right) \)

**Theorem 3.5.** Let \( \xi(t,r,\theta,\varphi) \) be a 4D random field in \( \mathbb{R} \times \mathbb{R}^3 \) which is homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables. If the spectrum of \( \xi(t,r,\theta,\varphi) \) is
bounded in time \( t \) and all its spectral measures are concentrated in \([-\omega, \omega] \times R_+\) then the mean square approximation by model (26) is such that

\[
E | \zeta(t, x) - \zeta_{N,M}(t, x)|^2 < \frac{5\pi r^3}{2M^2} \mu_3 + \frac{4}{\pi^2 (2N-1)} \left( 2\mu_0 - \frac{5\pi r^3}{2M^2} \mu_3 \right) < \varepsilon, \quad (36)
\]

and

\[
E | \zeta(t, x) - \zeta_{N,M}(t, x)|^2 < \frac{2^{2M+3} r^{2M+2} ((M+1)!)^2}{(2M+2)!(2M+3)!} \mu_{2M+2} + \frac{4}{\pi^2 (2N-1)} \left( 2\mu_0 - \frac{2^{2M+3} r^{2M+2} ((M+1)!)^2}{(2M+2)!(2M+3)!} \mu_{2M+2} \right) < \varepsilon, \quad (37)
\]

where \( \mu_k \) is evaluated by the formula (29).

\textbf{Proof.} We use mentioned above model (26) of random field \( \xi(t, r, \theta, \varphi) \) for construct mean square estimate of approximation of this random field by partial sum. For convenience we denote \( \Psi_\theta := c_{m,l} P_l^m(\cos \theta) \). Then we make the following transformations

\[
\xi(t, r, \theta, \varphi) - \xi_{N,M}(t, r, \theta, \varphi) = \sum_{m=0}^{\infty} \sum_{l=0}^{m} \Psi_\theta \left[ \zeta_{m,1}^l(t, r) \cos \varphi + \zeta_{m,2}^l(t, r) \sin \varphi \right] - \\
- \sum_{k=-N}^{N} \frac{\sin \omega \left( t - \frac{k\pi}{\omega} \right)}{\omega \left( t - \frac{k\pi}{\omega} \right)} \sum_{m=0}^{M} \sum_{l=0}^{m} \Psi_\theta \left[ \zeta_{m,1}^l \left( \frac{k\pi}{\omega}, r \right) \cos \varphi + \zeta_{m,2}^l \left( \frac{k\pi}{\omega}, r \right) \sin \varphi \right] + \\
\sum_{m=M+1}^{\infty} \sum_{l=0}^{m} \Psi_\theta \left[ \zeta_{m,1}^l(t, r) \cos \varphi + \zeta_{m,2}^l(t, r) \sin \varphi \right] + \\
+ \sum_{k=-\infty}^{\infty} \frac{\sin \omega \left( t - \frac{k\pi}{\omega} \right)}{\omega \left( t - \frac{k\pi}{\omega} \right)} \sum_{m=0}^{M} \sum_{l=0}^{m} \Psi_\theta \left[ \zeta_{m,1}^l \left( \frac{k\pi}{\omega}, r \right) \cos \varphi + \zeta_{m,2}^l \left( \frac{k\pi}{\omega}, r \right) \sin \varphi \right] -
\]
\[
- \sum_{k=-N}^{N} \sin \omega \left( \frac{t - k \pi}{\omega} \right) \sum_{m=0}^{M} \sum_{l=0}^{m} \Psi_{\theta} \left[ \zeta_{m,1}^{l} \left( \frac{k \pi}{\omega}, r \right) \cos \varphi + \zeta_{m,2}^{l} \left( \frac{k \pi}{\omega}, r \right) \sin \varphi \right] = \\
\sum_{m=N+1}^{\infty} \sum_{l=0}^{m} \Psi_{\theta} \left[ \zeta_{m,1}^{l} (t, r) \cos \varphi + \zeta_{m,2}^{l} (t, r) \sin \varphi \right] + \\
\sum_{k>N}^{\infty} \sum_{k<-N} \sin \omega \left( \frac{t - k \pi}{\omega} \right) \sum_{m=0}^{M} \sum_{l=0}^{m} \Psi_{\theta} \left[ \zeta_{m,1}^{l} \left( \frac{k \pi}{\omega}, r \right) \cos \varphi + \zeta_{m,2}^{l} \left( \frac{k \pi}{\omega}, r \right) \sin \varphi \right].
\]

Passing to the mathematical expectations of the square of the last expression and taking into account the mutual orthogonality of stochastic processes \( \zeta_{m,1}^{l} (t, r), \zeta_{m,2}^{l} (t, r) \) and \((a+b)^2 \leq 2a^2 + 2b^2\) we obtain

\[
E \left[ \xi(t, r, \varphi) - \xi_{N,M} (t, r, \vartheta, \varphi) \right]^2 \leq 2 \left. E \right| \sum_{m=0}^{M} \sum_{l=0}^{m} \Psi_{\theta} \left[ \zeta_{m,1}^{l} (t, r) \cos \varphi + \zeta_{m,2}^{l} (t, r) \sin \varphi \right] \right|^2 + \\
+ 2 \left. E \right| \sum_{k>N}^{\infty} \sum_{k<-N} \sin \omega \left( \frac{t - k \pi}{\omega} \right) \sum_{m=0}^{M} \sum_{l=0}^{m} \Psi_{\theta} \left[ \zeta_{m,1}^{l} \left( \frac{k \pi}{\omega}, r \right) \cos \varphi + \zeta_{m,2}^{l} \left( \frac{k \pi}{\omega}, r \right) \sin \varphi \right] \right|^2 \leq \\
\pi \sum_{m=0}^{M} \left( 2m + 1 \right) \bar{b}_{m} (0, r) + \pi \sum_{k+N}^{\infty} \sum_{k<-N} \sin \omega \left( \frac{t - k \pi}{\omega} \right) \sum_{m=0}^{M} \left( 2m + 1 \right) \bar{b}_{m} (0, r) = \\
\pi \sum_{m=M+1}^{\infty} \left( 2m + 1 \right) \bar{b}_{m} (0, r) +
\]
\[
\pi \sum_{k>N, k<-N} \left| \frac{\sin \omega \left( t - \frac{k \pi}{\omega} \right)}{\omega} \right|^2 \left( \sum_{m=0}^{\infty} (2m+1) \tilde{b}_m(0, r) - \sum_{m=M+1}^{\infty} (2m+1) \tilde{b}_m(0, r) \right).
\]

Then we use the following equality

\[
\sum_{k>N, k<-N} \left| \frac{\sin \omega \left( t - \frac{k \pi}{\omega} \right)}{\omega} \right|^2 = \sin^2(\omega t) \sum_{k>N, k<-N} \frac{1}{\pi^2 \left( \frac{\omega t}{\pi} - k \right)^2}.
\]

to estimate the second factor in the last term of inequality. We substitute the series

\[
\frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{\left( \frac{\omega t}{\pi} - k \right)^2} = \frac{1}{\sin^2(\omega t)}
\]

to the last expression. Then

\[
g_N \left( \frac{\omega t}{\pi} \right) = \sum_{k>N, k<-N} \left| \frac{\sin \omega \left( t - \frac{k \pi}{\omega} \right)}{\omega} \right|^2 = 1 - \sin^2(\omega t) \sum_{k=-N}^{N} \frac{1}{\pi^2 \left( \frac{\omega t}{\pi} - k \right)^2}.
\]

It is known from Olenko (2004) that

\[
\sup_{x \in x} g_N(x) = 1 - \frac{8}{\pi^2} \sum_{k=1}^{N} \frac{1}{(2k-1)^2}.
\]

Using the equality Olenko (2005)

\[
\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}
\]

we obtain the upper bound

\[
1 - \frac{8}{\pi^2} \sum_{k=1}^{N} \frac{1}{(2k-1)^2} \leq \frac{8}{\pi^2} \int_{N}^{\infty} \frac{dx}{(2x-1)^2} = \frac{4}{\pi^2 (2N-1)}
\]

since the function \( \frac{1}{(2x-1)^2} \) decreases for \( x > 1 \).
Now simplified mean square estimates for the approximation of a random field 
\( \xi(t, r, \theta, \varphi) = \xi(t, x) \) by its model (26) follows from the inequalities (21), (32), (38), the equality (20). These estimates are written as follows

\[
E\left| \xi(t, x) - \xi_{N,M}(t, x) \right|^2 < \frac{5\pi r^3}{2M^2} \mu_3 + \frac{4}{\pi^2 (2N-1)} \left( 2\mu_0 - \frac{5\pi r^3}{2M^2} \mu_3 \right) < \varepsilon,
\]

and

\[
E\left| \xi(t, x) - \xi_{N,M}(t, x) \right|^2 < \frac{2^{2M+3} r^{2M+2} ((M+1)!)^2}{(2M+2)!(2M+3)!} \frac{\mu_{2M+2}}{\mu_{2M+2}^2} + \frac{4}{\pi^2 (2N-1)} \left( 2\mu_0 - \frac{2^{2M+3} r^{2M+2} ((M+1)!)^2}{(2M+2)!(2M+3)!} \mu_{2M+2} \right) < \varepsilon.
\]

The second term of these estimates has the order of convergence \( O\left( \frac{1}{N} \right) \). The inequalities (36) and (37) hold true.

4. A procedure for the Statistical Simulation of a Random Field

The above Kotelnikov-Shannon expansion for 4D random field \( \xi(t, r, \theta, \varphi) \) in \( R \times R^3 \) which is time homogeneous and homogeneous isotropic with respect to the spatial variables can be used for the statistical simulation of such random field if the spectrum of the field is bounded with respect to \( t \) and if some statistical characteristics of the field are given.

Below we describe a procedure for the statistical simulation of realizations of Gaussian random fields \( \xi(t, r, \theta, \varphi) \) being homogeneous with respect to time and homogeneous isotropic with respect to the spatial variables whose spectrum is bounded in \( t \) which is based on the model (26) and estimates (27), (28), (33), (34), (36) and (37).

Recall that a random field in \( R \times R^3 \) which is time homogeneous and homogeneous isotropic with respect to the spatial variables can be written as in the (4). We use partial sums of (4) and partial sums of expansion (16) for the random field \( \xi(t, r, \theta, \varphi) \) to construct its model (26).

The partial sum (26) is taken as an approximation model of such a random field.
Using (26), the procedure for the statistical simulation of realizations of a Gaussian random field which is time homogeneous and homogeneous isotropic with respect to spatial variables can be stated as follows if its spectrum is bounded in $t$. The procedure is described below.

- We choose positive integer numbers $N$ and $M$ for the model (26) according to a prescribed accuracy $\varepsilon > 0$ by using one of the following inequalities (27), (28), (33), (34), (36) and (37).

- We generate values of Gaussian stochastic processes $\xi^l_{m,j}\left(\frac{k \pi}{\omega}, r\right)$ and for all $m, p = 0, 1, \ldots, M; k, q = -N, N; l, s = 0, 1, \ldots, m; i, j = 1, 2$ and a fixed $r$ such that:

  $$
  \xi^l_{m,j}\left(\frac{k \pi}{\omega}, r\right) = 0, \ E\xi^l_{m,j}\left(\frac{k \pi}{\omega}, r\right)\xi^s_{p,i}\left(\frac{q \pi}{\omega}, r\right) = \delta^l_i \delta^m_p \delta^s_j b_m \left(\frac{(k - q) \pi}{\omega}, r\right).
  $$

- We evaluate the expression in (26) at a given point $(t, r, \theta, \varphi) \in [-T, T] \times \mathbb{R}^3$, by substituting the numbers $N, M$ and values of Gaussian stochastic processes evaluated in steps 1 and 2.

- We check whether the realization of the stochastic random field $\xi(t, r, \theta, \varphi)$ generated in step 3 fits the data by testing the corresponding statistical characteristics.

5. Conclusions

This paper presents results of investigation of real valued and mean square continuous random fields which are homogeneous with respect to the time variable and homogeneous isotropic with respect to the 3D spatial variables in $\mathbb{R} \times \mathbb{R}^3$. Statistical simulation model, procedure and realizations of these real valued random fields with Gaussian distribution and a bounded spectrum are constructed. The mean square estimates of these random fields approximating by its models are found. By using of these estimates, models and procedure there are demonstrated possibility to generate on computer adequate realizations of such random field with wide-known examples of covariance functions for spatial-temporal data. These results continued investigations (set in work Vyzhva, 2011; Vyzhva et al., 2012) of modelling and generation method of noise seismogram implementations at flat observation area (Vyzhva, 2012; Vyzhva and Fedorenko, 2014) and seismograms from 3D observation area Vyzhva (2013).

References


http://dx.doi.org/10.1137/1104040


