A Class of Estimators of Finite Population Total using Two Auxiliary Variables

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Abstract

This paper proposes a class of estimators for population total \( Y \) using two auxiliary variables. In addition to many, it is identified that the proposed class of estimators includes the estimators due to Singh, M.P. (1967), Srivastava, S.K. (1967), Srivastava, V.K. (1974), Singh, H.P. (1986), Prasad, B. (1989) and Gandge et al. (1993). To the first degree of approximation, the bias and mean squared error of the proposed class of estimators have been obtained. The optimum condition is obtained in which the proposed class of estimators has minimum mean squared error. The superiority of the proposed class of estimators is also discussed. An empirical study is carried in support of the present study.

Keywords: Auxiliary variable; Study variable; Population total; Bias; Mean squared error; Empirical study

1. Introduction

It is well known that the suitable use of auxiliary information in probability sampling results in considerable reduction in variance of the estimator of population mean / total. The auxiliary information may be used either at the stage of designing or at the stage of estimation, depending upon the form in which such information is available for increase precision of the estimate of the population characteristics under study. If there is more than one auxiliary character, the problem remains as to how the entire information can be utilized in a better way. Multivariate ratio and regression methods of estimation, two way stratification etc.; provide some alternative solution to the problem.

Singh (1965, 1967) also suggested a method of using the two auxiliary variates by considering a

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ratio-cum-product estimator for estimating the population total of study variable. The descriptions of the estimators for the population total envisaged by Singh (1967) are given below.

For any sample design, let \( \hat{Y} \) be the unbiased estimator of the population totals \( Y \) of the main variable \( y \) and \( \hat{X}_1, \hat{X}_2 \) be the unbiased estimators of the population totals \( X_1, X_2 \) of two auxiliary variables \( x_1, x_2 \) respectively. Also let \( C_0, C_1 \) and \( C_2 \) be the coefficients of variation \( R \) and \( \rho \) be the correlation coefficients between \( (\hat{Y}, \hat{X}_1) \), \( (\hat{Y}, \hat{X}_2) \) and \( (\hat{X}_1, \hat{X}_2) \) respectively. Singh (1967) argued that the population totals \( X_1, X_2 \) can be used together on an analog of ratio (using the positively correlated variable in the form of ratio) and product estimators (using the negatively correlated variable in the form of product) and suggested the following three forms of the estimators for population total \( Y \).

(i) Ratio-cum-product-type estimator

\[
\hat{Y}_1 = Y \left( \frac{X_1}{X_1} \right) \left( \frac{\hat{X}_2}{X_2} \right) = \hat{Y}_R \left( \frac{\hat{X}_2}{X_2} \right),
\]

(1.1)

(ii) Ratio-type estimator

\[
\hat{Y}_2 = \hat{Y} \left( \frac{X_1}{X_1} \right) \left( \frac{X_2}{X_2} \right) = \hat{Y}_R \left( \frac{X_2}{X_2} \right),
\]

(1.2)

(iii) Product-type estimator

\[
\hat{Y}_3 = \hat{Y} \left( \frac{\hat{X}_1}{X_1} \right) \left( \frac{\hat{X}_2}{X_2} \right) = \hat{Y}_P \left( \frac{\hat{X}_2}{X_2} \right),
\]

(1.3)

where \( \hat{Y}_R \) and \( \hat{Y}_P \) are respectively the usual ratio and product estimators for the population total \( Y \) defined by

\[
\hat{Y}_R = Y \left( \frac{\hat{X}_1}{X_1} \right),
\]

(1.4)

\[
\hat{Y}_P = Y \left( \frac{\hat{X}_1}{X_1} \right).
\]

(1.5)

To the first degree of approximation, the biases and mean squared errors (MSEs) of the estimators \( \hat{Y}_R, \hat{Y}_P, \hat{Y}_1, \hat{Y}_2 \) and \( \hat{Y}_3 \) are respectively given by

\[
B(\hat{Y}_R) = Y (C_1^2 - \rho_{01} C_0 C_1),
\]

(1.6)

\[
B(\hat{Y}_P) = Y \rho_{01} C_0 C_1,
\]

(1.7)

\[
B(\hat{Y}_1) = B(\hat{Y}_R) + Y C_1^2 C,
\]

(1.8)

\[
B(\hat{Y}_2) = B(\hat{Y}_R) + Y C_2^2 (1 - C),
\]

(1.9)

\[
B(\hat{Y}_3) = B(\hat{Y}_R) + Y C_1^2 C,
\]

(1.10)
From (1.11), (1.12) and (1.13) we have that
\[ \text{MSE} \left( \hat{Y} \right) = Y^2 \left( C_0^2 + C_1^2 - 2 \rho_0 C_0 C_1 \right), \]
where \( C = \left\{ \rho_2 \left( \frac{C_0}{C_2} \right) - \rho_1 \left( \frac{C_1}{C_2} \right) \right\}, \)
and \( C^* = \left\{ \rho_2 \left( \frac{C_0}{C_2} \right) + \rho_1 \left( \frac{C_1}{C_2} \right) \right\}. \)

Thus it is observed from (1.19) and (1.20) that the ratio estimator \( \hat{Y}_R \) is more efficient than the conventional unbiased estimator \( \hat{Y} \) if \( K > 1/2 \),
and the product estimator \( \hat{Y}_p \) is more efficient than the usual unbiased estimator \( \hat{Y} \) if \( K < -1/2 \).

Now from (1.11), (1.14), (1.15) and (1.16) we have
\[ \text{MSE} \left( \hat{Y}_R \right) - \text{MSE} \left( \hat{Y} \right) = Y^2 \left[ C_1^2 (2K-1) - C_2^2 (1+2C) \right], \]
\[ \text{MSE} \left( \hat{Y}_p \right) - \text{MSE} \left( \hat{Y} \right) = Y^2 \left[ C_1^2 (2K-1) + C_2^2 (2C-1) \right], \]
\[ \text{MSE} \left( \hat{Y}_p \right) - \text{MSE} \left( \hat{Y}_p \right) = -Y^2 \left[ C_1^2 (2K-1) + C_2^2 (2C^* + 1) \right]. \]

From (1.21), (1.22) and (1.23) we have that
(i) \( \text{MSE} \left( \hat{Y} \right) - \text{MSE} \left( \hat{Y}_R \right) > 0 \)
if \( C_1^2 (2K-1) > C_2^2 (1+2C) \),
(ii) \( \text{MSE} \left( \hat{Y} \right) - \text{MSE} \left( \hat{Y}_R \right) > 0 \)
if \( C_1^2 (2K-1) > C_2^2 (1+2C) \),
It follows from (1.24), (1.25) and (1.26) that the estimators \( \hat{Y}_1 \), \( \hat{Y}_2 \) and \( \hat{Y}_3 \) are better than the conventional unbiased estimator \( \hat{Y} \) as long as the conditions (1.24), (1.25) and (1.26) are satisfied.

Next, from (1.13) and (1.16) we have
\[
MSH(\hat{Y})R < MSH(\hat{Y}_3)
\]
if \( C < -1/2 \). (1.29)

Thus from (1.27), (1.28) and (1.29) it is observed that the estimator \( \hat{Y}_1 \), \( \hat{Y}_2 \) and \( \hat{Y}_3 \) are more efficient than \( \hat{Y}_R \) as long as conditions given in (1.26), (1.27) and (1.28) are respectively satisfied.

- **Ray and Singh (1981) estimator**

Ray and Singh (1981) proposed an alternative estimator for the population total \( Y \) as
\[
\hat{Y}_{R\theta} = (1+\theta)\hat{Y}_R - \theta \hat{Y}_1,
\]
where \( \theta \) is a scalar constant which can be suitably chosen (depending on the value \( C_0 \) or \( C_1 \) where \( C < C_0 (\neq -1/2) \) or \( C > C_1 (\neq 1/2) \)). It may be observed that \( \hat{Y}_{R\theta} \) reduces to \( \hat{Y}_1 \) and \( \hat{Y}_2 \) respectively for \( \theta = 0 \) and \( \theta = -1 \).

To the first degree approximation, the bias and the mean squared error (MSE) of the estimator \( \hat{Y}_{R\theta} \) due to Ray and Singh (1981) are respectively given by
\[
B(\hat{Y}_{R\theta}) = YC_0^2(1-K) - \theta C_2^2 C,
\]
\[
MSH(\hat{Y}_{R\theta}) = MSH(\hat{Y})R + Y^2 \theta C_2^2 (\theta - 2C),
\]
where \( MSH(\hat{Y}_R) \) is given by (1.12).

The optimum value of \( \theta \) that minimizes the MSE of \( \hat{Y}_{R\theta} \) is
\[
\theta = C.
\]

Substituting \( \theta = C \) in (1.32) we get the minimum MSE of \( \hat{Y}_{R\theta} \) as
\[
\min MSE(\hat{Y}_{R\theta}) = MSH(\hat{Y})R - Y^2 C_2^2 C^2,
\]
which clearly indicates that the estimator \( \hat{Y}_{R_0} \) due to Ray and Singh (1981) has smaller MSE than the ratio estimator \( \hat{Y}_R \) at the optimum condition \( \theta = C \).

- **Abu-Dayyeh et al. (2003) Class of Estimator**

Abu-Dayyeh et al.'s (2003) estimator for population total \( Y(=NY) \) is defined by

\[
\hat{Y}_a = \hat{Y} \left( \frac{X_1}{X} \right)^{\alpha_1} \left( \frac{X_2}{X} \right)^{\alpha_2},
\]

where \((\alpha_1, \alpha_2)\) are suitable chosen constants.

To the first degree of approximation, the mean squared error of \( \hat{Y}_a \) is given by

\[
MSR(\hat{Y}_a) = \hat{Y} [C_0^2 + \alpha_1^2 C_1^2 + \alpha_2^2 C_2^2 - 2\alpha_1 \alpha_2 C_1^2 K - 2\alpha_1 \alpha_2 C_2^2 C_{(a)}].
\]

where \( C_{(a)} = \left\{ \rho_{02} \left( \frac{C_0}{C_2} \right) - \alpha_1 \alpha_2 \left( \frac{C_1}{C_2} \right) \right\} \).

The \( MSR(\hat{Y}_a) \) is minimized for

\[
\alpha_1 = \frac{(\rho_{01} - \rho_{02} \rho_{12}) C_0}{(1 - \rho_{12}^2) C_1} = \alpha_{10} (say) \]

\[
\alpha_2 = \frac{(\rho_{02} - \rho_{01} \rho_{12}) C_0}{(1 - \rho_{12}^2) C_2} = \alpha_{20} (say)
\]

Thus the minimum MSE of \( \hat{Y}_a \) is given by

\[
\min MSR(\hat{Y}_a) = Y^2 C_0^2 (1 - R_{112}^2),
\]

where \( R_{112}^2 = \frac{(\rho_{01}^2 + \rho_{02}^2 - 2\rho_{01} \rho_{02} \rho_{12})}{1 - \rho_{12}^2} \)

is the square of multiple correlation coefficient between study variable \( y \) and the joint effect of auxiliary variables \((x_1, x_2)\).

- **Kadilar and Cingi (2005) Class of Estimator**

Kadilar and Cingi's (2005) estimator for population total \( Y \) is given by

\[
\hat{Y}_{KC} = \hat{Y} \left( \frac{X_1}{X} \right)^{\alpha} \left( \frac{X_2}{X} \right)^{\alpha} + b_1 (X_1 - \hat{X}_1) + b_2 (X_2 - \hat{X}_2),
\]

where \((\alpha, \alpha)\) are same as defined earlier, and \((b_1, b_2)\) are the sample regression coefficients of \( \hat{Y} \) on \( \hat{X}_1 \) and \( \hat{Y} \) on \( \hat{X}_2 \) respectively.
• **Perri (2007) Class of Estimator**

Perri’s (2007) estimators for population total $Y$ are defined by

\[
\hat{Y}_{P(1)} = \hat{Y} \left( \hat{X}_2 + \alpha_2(X_2 - \hat{X}_2) \right) \left( \frac{X_1}{X_2} \right) \]

\[
\hat{Y}_{P(2)} = \hat{Y} \left( \frac{X_1}{X_2 + \alpha_2(X_2 - X_2)} \right) \left( \frac{X_2}{X_1} \right) \]

\[
\hat{Y}_{P(3)} = \hat{Y} \left( \frac{X_1}{X_1 + \alpha_2(X_1 - X_1)} \right) \left( \frac{X_2}{X_2 + \alpha_2(X_2 - X_2)} \right) \]

where $(\alpha_1, \alpha_2)$ are same as defined earlier.

• **Singh et al. (2009) Class of Estimator**

Singh et al.’s (2009) estimator for population total $Y$ is given by

\[
\hat{Y}_S = \hat{Y} \exp \left( \frac{\alpha_1(X_1 - \hat{X}_1)}{(X_1 + \hat{X}_1)} \right) \exp \left( \frac{\alpha_2(X_2 - \hat{X}_2)}{(X_2 + \hat{X}_2)} \right),
\]

where $(\alpha_1, \alpha_2)$ are same as defined earlier.

It can be easily shown to the first order of approximation that the common minimum $MSE$ of the estimator $\hat{Y}_{KC}, \hat{Y}_{P(j)}, j=1,2,3; \text{ and } \hat{Y}_S$ is:

\[
\min MSE \hat{Y}_{KC} \text{ or } \hat{Y}_{P(j)} \text{ or } \hat{Y}_S = Y^2 C_0^2 (1 - R_{012}^2)
\]

which equals to the minimum $MSE$ of the Abu-Dayyeh et al.’s (2003) estimator $\hat{Y}_\alpha$ and the difference estimator based on auxiliary variable $(x_1, x_2)$ defined by

\[
\hat{Y}_\alpha = \hat{Y} + \alpha_1(X_1 - \hat{X}_1) + \alpha_2(X_2 - \hat{X}_2),
\]

where $(\alpha_1, \alpha_2)$ are same as defined earlier.

• **Regression Estimator**

The regression estimator for population total $Y$ is given by

\[
\hat{Y}_l = \hat{Y} + b_1(X_1 - \hat{X}_1) + b_2(X_2 - \hat{X}_2)
\]

which can be obtained from Kadilar and Cingi’s (2005) class of estimators by putting $(\alpha_1, \alpha_2) = (0, 0)$.

To the first degree of approximation, the $MSE$ of $\hat{Y}_l$ is given by

\[
MSE(\hat{Y}) = Y^2 C_0^2 (1 - \rho_{01}^2 - \rho_{02}^2 + 2 \rho_{0j} \rho_{02} \rho_{2j}) \\
= Y^2 C_0^2 [1 - R_{012}^2 (1 - \rho_{12}^2)]
\]
From (1.34), (1.38), (1.45) and (1.48) we have

\[
MSR(\hat{Y}_{R\theta}) - \min \, MSR(\hat{Y}_j) = Y^2 C_1 (1 - \rho_{12}^2) + (\rho_{02} - \rho_{01}) C_0^2 \frac{1}{(1 - \rho_{12}^2)} \geq 0
\]

(1.49)

\[
MSR(\hat{Y}_j) - \min \, MSR(\hat{Y}_j) = Y^2 C_0^2 \rho_{12}^2 R_{012}^2 \geq 0,
\]

(1.50)

where \( j = a, KC, P(1), P(2), P(3), S \) and \( \alpha \). It follows from (1.38), (1.45), (1.49) and (1.50) that the classes of estimators \( \hat{Y}_a, \hat{Y}_{KC}, \hat{Y}_{P(j)}, j = 1, 2, 3 \); \( \hat{Y}_S \) and the difference estimator \( \hat{Y}_\alpha \) are equally efficient but more efficient than the estimator \( \hat{Y}_{R\theta} \) due to Ray and Singh (1981) and regression estimator \( \hat{Y}_l \).

When the population totals \( X_1 \) and \( X_2 \) of auxiliary variables \( x_1 \) and \( x_2 \) respectively are known, Singh and Tailor (2005) and Subramani and Prabavathy (2014) have further suggested the various estimators for population mean \( \bar{Y} \) (or total \( Y \)) based on correlation coefficient \( \rho_{12} \) between the auxiliary variables \( x_1 \) and \( x_2 \) and population coefficients of kurtosis \( \beta_2(x_1) \) and \( \beta_2(x_2) \) of the auxiliary variable \( x_1 \) and \( x_2 \) respectively; and studied their properties under large sample approximation.

It should be mentioned here that to the first degree of approximation, the efficiencies of the estimators due to Singh and Tailor (2005) and Subramani and Prabavathy (2014) can be at the most equal to the estimators \( \hat{Y}_a, \hat{Y}_{KC}, \hat{Y}_{P(j)}, j = 1, 2, 3; \hat{Y}_S \) and the difference estimator \( \hat{Y}_\alpha \) at their optimum conditions. These lead authors to investigate a class of estimators better than the estimators reported in this section. This is the principal objective of the present paper. To fulfill this objective, we have suggested a class of estimators better than the conventional unbiased estimator \( \hat{Y} \), ratio estimator \( \hat{Y}_R \), Singh (1967) estimator \( \hat{Y}_1 \), Ray and Singh (1981) estimator \( \hat{Y}_{R\theta} \), Abu-Dayyeh et al.(2003) estimator \( \hat{Y}_a \), Kadilar and Cingi (2005) estimator \( \hat{Y}_{KC} \), Perri(2007) estimators \( \hat{Y}_{P(j)}, j = 1, 2, 3; \) difference estimator \( \hat{Y}_a \), regression estimator \( \hat{Y}_l \) and other estimators.

2. Suggested Class of Estimators for Population Total

We define the following class of estimators for population total \( Y \) as

\[
d = \left[ w_1 \left( \frac{X_1}{\hat{X}_1} \right)^{q_1} + w_2 \left( X_1 - \hat{X}_1 \right) \left( \frac{X_2}{\hat{X}_2} \right)^{q_2} \right],
\]

(2.1)
(α₁, α₂) being suitably chosen scalars may take values (-1, 0, +1), and (w₁, w₂) are suitably chosen constants such that the MSE of ‘d’ is minimum. We note that for suitable values of (w₁, w₂, α₁, α₂) the estimator d reduces to the following set of estimators:

(i) for (w₁, w₂, α₁, α₂) = (1, 0, 0, 0)

\[ d \rightarrow \hat{Y} \]

(ii) for (w₁, w₂, α₁, α₂) = (1, 0, 1, 0)

\[ d \rightarrow \hat{Y}_1 = \hat{Y}\left(\frac{X_1}{X_1}\right) \]

(iii) for (w₁, w₂, α₁, α₂) = (1, 0, 1, -1)

\[ d \rightarrow \hat{Y}_2 = \hat{Y}\left(\frac{X_2}{X_2}\right) \]

(iv) for (w₁, w₂, α₁, α₂) = (1, 0, 1, 1)

\[ d \rightarrow \hat{Y}_3 = \hat{Y}\left(\frac{X_1}{X_1}\right) \]

(v) for (w₁, w₂, α₁, α₂) = (1, 0, -1, -1)

\[ d \rightarrow \hat{Y}_4 = \hat{Y}\left(\frac{X_1}{X_1}\right) \]

(vi) for (w₁, w₂, α₁, α₂) = (1, 0, 1, 0)

\[ d \rightarrow \hat{Y}_5 = \hat{Y}\left(\frac{X_1}{X_1}\right) \]

which is due to Srivastava (1967).

To obtain the bias and MSE of the proposed class of estimators ‘d’, we write

\[ \hat{Y} = Y(1+e_i), \hat{X}_1 = X_1(1+e_1), \hat{X}_2 = X_2(1+e_2) \]

such that

\[ E(e_i) = 0 \quad \forall i = 0, 1, 2, \]

\[ E(e_i^2) = C_i^2 \quad \forall i = 0, 1, 2, \]

\[ E(e_0e_1) = \rho_{01}C_0C_1, \quad E(e_0e_2) = \rho_{02}C_0C_2 \quad \text{and} \quad E(e_1e_2) = \rho_{12}C_1C_2. \]

Expressing the class of estimators ‘d’ in terms of eᵢ’s, we have
\[ d = \left[ \frac{X_1}{X_1(1+e_1)} \right]^{\alpha_1} + \frac{X_2}{X_2(1+e_2)} \right]^{\alpha_2} \]

\[ = \left[ \frac{X_1}{X_1(1+e_1)} \right]^{\alpha_1} - \frac{X_1e_1}{(1+e_2)}^{\alpha_2}. \]  

(2.2)

We assume \(|e_i| < 1, i = 1, 2\); so that \((1+e_i)^{-\alpha_i}\), \(i = 1, 2\) is expandable. Expanding the right hand side of (2.2) we have

\[ d' = \left[ \frac{X_1}{X_1(1+e_1)} \right]^{\alpha_1} + \frac{\alpha_1(\alpha_1 + 1)}{2}e_1^2 \ldots \right]_{e=1}^{2} \right] - w_2X_1e_1 \right] \]

\[ = \left[ \frac{X_1}{X_1(1+e_1)} \right]^{\alpha_1} - \frac{X_1e_1}{(1+e_2)}^{\alpha_2} - \frac{\alpha_2(\alpha_2 + 1)}{2}e_2^2 \ldots \right] \]

Multiplying out and neglecting terms of \( e_i \)'s having power greater than two we have

\[ d' \approx \left[ \frac{X_1}{X_1(1+e_1)} \right]^{\alpha_1} - \frac{\alpha_1(\alpha_1 + 1)}{2}e_1^2 \ldots \right] - w_2X_1e_1 \right] \]

\[ (d-Y) \approx \left[ \frac{X_1}{X_1(1+e_1)} \right]^{\alpha_1} - \frac{\alpha_1(\alpha_1 + 1)}{2}e_1^2 \ldots \right] - w_2X_1e_1 \right] \]

where \( R_i = \frac{X_1}{Y} \). 

Taking expectation of both sides of (2.3) we get the bias of the estimator \( 'd' \) to the first degree of approximation as

\[ B(d) = \left[ \frac{\alpha_1(\alpha_1 + 1)}{2}C_1^2 - \alpha_1\alpha_0C_0C_1 - \alpha_2\alpha_0^2C_0C_2 + \alpha_4\alpha_2\alpha_1C_1C_2 \right] \]

\[ + \frac{\alpha_2(\alpha_2 + 1)}{2}C_2^2 \right] \]

(2.4)

Squaring both sides of (2.3) and neglecting terms of \( e_i \)'s having power greater than two we have

\[ (d-Y)^2 \approx Y^2[1 + w_1^2 + 2e_0 - 2\alpha_1e_1 - 2\alpha_2e_2 + e_0^2 - 4\alpha_1e_0e_1 - 4\alpha_2e_0e_2 + 4\alpha_2\alpha_2e_2 \]

\[ + \frac{\alpha_1(\alpha_1 + 1)}{2}e_1^2 + \frac{\alpha_2(\alpha_2 + 1)}{2}e_2^2 \right] \]

\[ + w_2R^2e_2^2 + 2w_1w_2R \{ \alpha_1e_1 - e_0e_1 + \alpha_2(\alpha_2 + 1)e_2 \} + \{ \alpha_1(\alpha_1 + 1)e_1^2 + \alpha_2(\alpha_2 + 1)e_2^2 \} - 2w_2R(\alpha_2e_2 - e_0) \]  

(2.5)

Taking expectation of both sides of (2.5) we get the \( MSE \) of the proposed estimator \( 'd' \) to the first degree of approximation as
\begin{align}
MSR(d) &= Y^2[1+w_1^2A_1+w_2^2A_2+2w_1w_2A_3-2w_1A_4-2w_2A_5], \\
\text{where} \quad C_{(a)} &= \left\{ \alpha_0 \left( \frac{C_0}{C_2} \right) - \alpha_1 \left( \frac{C_1}{C_2} \right) \right\}, \\
A_1 &= [1+C_0^2 + \alpha_4 C_1^2 (2\alpha_2 - 4K + 1) + \alpha_5 C_2^2 (2\alpha_2 - 4C_{(a)} + 1)], \\
A_2 &= R_1^2 C_1^2, \\
A_3 &= R_1^2 \{ C_1^2 (\alpha_4 - K) + 2\alpha_2 \alpha_1 C_1^2 C_2 \}, \\
A_4 &= \left[ 1 + C_1^2 \left( \frac{\alpha_4 + K}{2} + C_2^2 \left( \frac{\alpha_2 + 1}{2} - C_{(a)} \right) \right) \right], \\
A_5 &= \left[ R_1^2 \alpha_2 \alpha_1 C_1 C_2 \right].
\end{align}

Differentiating (2.6) partially with respect to \( w_1 \) and \( w_2 \) and equating them to zero, we have
\begin{align}
\begin{bmatrix}
A_1 & A_3 \\
A_3 & A_2
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
&=
\begin{bmatrix}
A_4 \\
A_5
\end{bmatrix}.
\end{align}

After simplifying (2.7) we get the optimum values of \( w_1 \) and \( w_2 \) as
\begin{align}
w_{10} &= \left( \frac{A_2 A_4 - A_3 A_5}{A_2 A_4 - A_3^2} \right), \\
w_{20} &= \left( \frac{A_3 A_5 - A_4 A_4}{A_2 A_4 - A_3^2} \right). \tag{2.8}
\end{align}

Inserting \(( w_{10}, w_{20} )\) in place of \(( w_1, w_2 )\) respectively in (2.6) we get the resulting minimum \( MSE \) of the estimator \( 'd' \) as
\begin{align}
\min MSR(d) &= Y^2 \left[ 1 - \frac{A_2 A_4^2 - 2A_3 A_4 A_5 + A_4^2}{A_2 A_4 - A_3^2} \right]. \tag{2.9}
\end{align}

Thus we state the following theorem.

**Theorem 2.1:** To the first order of approximation,
\begin{align}
MSR(d) &\geq Y^2 \left[ 1 - \frac{A_2 A_4^2 - 2A_3 A_4 A_5 + A_4^2}{A_2 A_4 - A_3^2} \right], \tag{2.10}
\end{align}

with equality holding if
\begin{align}
w_1 &= w_{10}, \\
w_2 &= w_{20},
\end{align}
where \( w_{i0} \)'s \ ((i = 1,2) \) are given by (2.8).

**Special Case I:** If we set \( w_2 = 0 \) in \( 'd' \) we get a class of estimators for population total \( Y \) as
\[ d^* = w_i \hat{Y} \left( \frac{X_1}{X} \right)^{\alpha_1} \left( \frac{X_2}{X} \right)^{\alpha_2}. \]  

(2.11)

Inserting \( w_2 = 0 \) in \( MSE(d) \) at (2.6) we get the MSE of \( d^* \) to the first degree of approximation as 
\[ MSE(d^*) = Y^2 \left[ 1 + w_1^2 A_1 - 2 w_1 A_4 \right], \]  

(2.12)

which is minimized for 
\[ w_i = \frac{A_4}{A_1} = w_i' \text{(say)}. \]  

(2.13)

Thus the resulting minimum MSE of \( d^* \) is given by 
\[ \min MSE(d^*) = Y^2 \left( 1 - \frac{A_4^2}{A_1} \right). \]  

(2.14)

Putting \((w_1, w_2) = (1, 0)\) in (2.1) \(d^*\) we get a class of estimators for population total \( Y \) as 
\[ \hat{Y}_a = \hat{Y} \left( \frac{X_1}{X} \right)^{\alpha_1} \left( \frac{X_2}{X} \right)^{\alpha_2} \]  

(2.15)

which is due to Abu-Dayyeh et al. (2003).

Substituting \((w_1, w_2) = (1, 0)\) in MSE of \( d \) at (2.6) we get the MSE of Abu-Dayyeh et al. (2003) estimator \( \hat{Y}_a \) to the first degree of approximation as 
\[ MSE(\hat{Y}_a) = Y^2 \left[ 1 + A_1 - 2 A_4 \right] \]  

(2.16)

\[ = Y^2 \left[ C_0^2 + \alpha_1^2 \alpha_4 C_1 + \alpha_2^2 C_2 - 2 \alpha_1 A_4 C_1^2 - 2 \alpha_2 K_{02} C_2^2 - 2 \alpha_4 \alpha_2 \rho_1 C_2 C_4 \right], \]  

(2.17)

where \( K_{02} = \rho_{02}(C_0 / C_2) \).

From (2.9), (2.14) and (2.16) we have 
\[ \min MSE(d^*) - \min MSE(d) = Y^2 \frac{(A_4 A_5 - A_4 A_4)^2}{A_1 (A_5 A_2 - A_5^2)} \geq 0. \]  

(2.18)

\[ \min MSE(\hat{Y}_a) - \min MSE(d^*) = Y^2 \frac{(A_1 - A_4)^2}{A_1} \geq 0. \]  

(2.19)

Thus from (2.18) and (2.19) we have the following inequality: 
\[ \min MSE(d) \leq \min MSE(d^*) \leq \min MSE(\hat{Y}_a). \]  

(2.20)

It follows from (2.20) that the proposed class of estimators \( d \) is more efficient than the class of estimators \( d^* \) and \( \hat{Y}_{R_0} \), Abu-Dayyeh et al. (2003) estimator \( \hat{Y}_a \) irrespective of the values of scalars \((\alpha_1, \alpha_2)\).
Hence the classes of estimators \((d, d')\) are also better than Singh (1967) estimator \(\hat{Y}_1\), Ray and Singh (1981) estimator \(\hat{Y}_{R0}\), Abu-Dayyeh et al (2003) estimator \(\hat{Y}_a\), Kadilar and Cingi (2005) estimator \(\hat{Y}_{KC}\), Perri(2007) estimators \(\hat{Y}_{P(j)}\), \(j=1,2,3\); and Singh et al’s (2009) estimator \(\hat{Y}_5\).

**Special Case II:** For \((\alpha_1, \alpha_2) = (1, -1)\) in (2.1), the class of estimators ‘\(d\)’ reduces to the estimator

\[
d_1 = \left[ w_1 \hat{Y} \left( \frac{X_1}{X_1} \right) + w_2 (X_1 - \hat{X}_1) \right] \left( \frac{\hat{X}_2}{X_2} \right). \tag{2.21}
\]

Putting \((\alpha_1, \alpha_2) = (1, -1)\) in (2.4) and (2.6) we get the bias and \(MSE\) of \(d_1\) to the first degree of approximation respectively as

\[
B(d_1) = Y \left[ w_1 (1 + C_1^2 (1 - K) + C_2^2 C + w_2 R_1 \rho_{12} C_1 C_2 - 1) \right] \tag{2.22}
\]

\[
MSE(d_1) = Y^2 \left[ 1 + w_1^2 \alpha_1 + w_2^2 \alpha_2 + 2 w_1 w_2 \alpha_3 - 2 w_1 \alpha_4 - 2 w_2 \alpha_5 \right], \tag{2.23}
\]

where

\[
\alpha_1 = 1 + C_1^2 + C_2^2 (3 - 4 K) + 4 C_2^2 C,
\]

\[
\alpha_2 = R_1^2 C_1^2,
\]

\[
\alpha_3 = R_1 C_1 (1 - K) - 2 R_{12} C_1 C_2,
\]

\[
\alpha_4 = 1 + C_1^2 (1 - K) + C_2^2 C,
\]

\[
\alpha_5 = R_1 \rho_{12} C_1 C_2.
\]

Differentiating (2.23) partially with respect to \(w_1\) and \(w_2\) and equating to zero we have the equation;

\[
\begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} a_4 \\ -a_5 \end{bmatrix}. \tag{2.24}
\]

After simplifying (2.24) we get the optimum values of \(w_1\) and \(w_2\) as

\[
w_1 = \frac{a_2 a_4 + a_3 a_5}{a_4 a_2 - a_3^2} = w_{10}^* \text{(say)}
\]

\[
w_2 = \frac{-a_3 a_4 + a_3 a_4}{a_4 a_2 - a_3^2} = w_{20}^* \text{(say)} \tag{2.25}
\]

Substitution of (2.25) in (2.23) yields the minimum \(MSE\) of the estimator ‘\(d_1\)’ as

\[
\min MSE(d_1) = Y^2 \left[ 1 - \frac{a_2 a_4^2 - 2 a_3 a_4 a_5 + a_5 a_2^2}{a_4 a_2 - a_3^2} \right]. \tag{2.26}
\]

If we set \(w_2 = 0\) in (2.21) we get class of estimators for population total \(Y\) as

\[
d_{1(1)} = w_1 \hat{Y} \left( \frac{X_1}{X_1} \right) \left( \frac{\hat{X}_2}{X_2} \right). \tag{2.27}
\]
Inserting $w_2 = 0$ in (2.12) and (2.23) we get the bias and $MSE$ of $d_{\text{i}(l)}$ to the first degree of approximation respectively as

\begin{align}
B(d_{\text{i}(l)}) &= Y[w_1 \{1 + C_1^2 (1 - K) + C_2^2 C]\} - 1], \\
MSE(d_{\text{i}(l)}) &= Y^2 [1 + w_1^2 a_i - 2 w_i a_4].
\end{align}

(2.28) \hspace{1cm} (2.29)

The $MSE$ of $d_{\text{i}(l)}$ at (2.29) is minimized for

$$w_i = \frac{a_4}{a_l} = w_{i(\text{opt})}.$$ \hspace{1cm} (2.30)

Thus the resulting minimum $MSE$ of $d_{\text{i}(l)}$ is given by

$$\min MSE(d_{\text{i}(l)}) = Y^2 \left[ 1 - \frac{a_4^2}{a_l^2} \right].$$ \hspace{1cm} (2.31)

Further, if we put $w_i = 1$ in (2.27) we get the ratio-cum-product estimator due to Singh (1967) for population total $Y$ as

$$d_{\text{r}(l)} = \hat{Y} \left( \frac{X_1}{X_1} \right) \left( \frac{X_2}{X_2} \right),$$ \hspace{1cm} (2.32)

$$= \hat{Y}_i.$$

Putting $w_i = 1$ in (2.28) and (2.29) we get the bias and $MSE$ of $d_{\text{r}(l)} = \hat{Y}_i$ to the first degree of approximation respectively as

\begin{align}
B(\hat{Y}_i) &= B(d_{\text{r}(l)}) = Y' C_2^2 (1 - K) + C_2^2 C', \\
MSE(\hat{Y}_i) &= MSE(d_{\text{r}(l)}) = Y^2 [1 + a_i - 2 a_4].
\end{align}

(2.33) \hspace{1cm} (2.34)

If we do not use information on auxiliary variable $X_2$, then the estimator $d_{\text{i}(l)}$ at (2.17) reduces to the estimator

$$d_{\text{i}(l)\text{1}} = w_1 \hat{Y} \left( \frac{X_1}{X_1} \right) = w_1 \hat{Y}_R,$$ \hspace{1cm} (2.35)

which is due to Srivastava (1974) and Prasad (1989).

The bias and $MSE$ of $d_{\text{i}(l)\text{1}}$ to the first degree of approximation are respectively given by

\begin{align}
B(d_{\text{i}(l)\text{1}}) &= Y[ w_1 \{1 + C_1^2 (1 - K)\} - 1] \\
MSE(d_{\text{i}(l)\text{1}}) &= Y^2 [1 + w_1^2 a_i' - 2 w_i a_4'],
\end{align}

(2.36) \hspace{1cm} (2.37)

where $a_i' = [1 + C_0^2 + C_1^2 (3 - 4 K)]$, $a_4' = [1 + C_1^2 (1 - K)]$.

The $MSE$ of $d_{\text{i}(l)\text{1}}$ is minimized for

$$w_{i(\text{opt})} = \frac{a_4'}{a_l}.$$ \hspace{1cm} (2.38)

Thus the resulting minimum $MSE$ of $d_{\text{i}(l)\text{1}}$ is given by
\[ \min MSE(d_{(1)}) = Y^2 \left[ 1 - \frac{a_4^2}{a_1^2} \right]. \]  

(2.39)

If we set \( w_1 = 1 \) in (2.37), we get the MSE of the usual ratio estimators \( \hat{Y}_R \) as

\[ MSE(\hat{Y}_R) = Y^2 \left[ 1 + a_1^* - 2a_4^* \right], \]

\[ = Y^2 \left( C_0^2 + C_1^2 - 2\rho_1 C_0 C_1 \right), \]

which is the same as given in (1.22).

From (1.22) and (2.39) we have

\[ MSE(\hat{Y}_R) - \min MSE(d_{(1)}) = Y^2 \left( \frac{a_1^* - a_4^*}{a_1} \right)^2, \]

which is always positive.

Thus the class of estimators \( d_{(1)i} \) due to Srivastava (1974) is more efficient than the usual ratio estimator \( \hat{Y}_R \).

Further, if the information on auxiliary variable \( x_1 \) is not available, then the estimator \( d_{(1)} \) at (2.27) reduces to the estimator

\[ d_{(1)2} = w_1 \hat{X}_2 \left( \frac{\hat{X}_2}{X_2} \right), \]

(2.40)

which is due to Singh (1986) and Gandge et al. (1993).

The bias and MSE of \( d_{(1)2} \) to the first degree of approximation are respectively given by

\[ B(d_{(1)2}) = Y \left[ w_1 \{ 1 + C_2^2 K^* \} - 1 \right] \]

(2.41)

\[ MSE(d_{(1)2}) = Y^2 \left[ 1 + w_1^2 a_1^* - 2w_1 a_4^* \right], \]

(2.42)

where \( a_1^* = [1 + C_0^2 + 4C_2^2 K_{02}] \),

\( a_4^* = [1 + C_2^2 K_{02}] \) and \( K_{02} = \rho_{02} (C_0 / C_2) \).

The MSE of \( d_{(1)2} \) at (2.42) is minimized for

\[ w_{(1)opt}^* = \frac{a_4^*}{a_1^*}. \]

(2.43)

Thus the resulting minimum MSE of \( d_{(1)2} \) is given by

\[ \min MSE(d_{(1)2}) = Y^2 \left[ 1 - \frac{a_4^*}{a_1^*} \right]. \]

(2.44)

3. Efficiency Comparison

From (2.31) and (2.32) we have
\[ \text{MSR} \left( \hat{Y}_i \right) - \min \text{MSR} \left( d_{i(i)} \right) = Y^2 \left( \frac{a_i - a_i'}{a_i'} \right)^2 \geq 0, \]  
(3.1)

which shows that the proposed sub class of estimators \( d_{i(i)} \) is better than the Singh (1967) ratio-cum-product estimator \( \hat{Y}_1 \).

From (2.26) and (2.27) we have
\[
\min \text{MSR}(d_{i(i)}) - \min \text{MSR}(d_i) = Y^2 \left[ \frac{a_i a_i^2 - 2a_i a_i a_i' + a_i a_i'}{a_i a_i' - a_i'^2} \right] - \frac{a_i^2}{a_i'} \]
\[ = Y^2 \left( \frac{a_i a_i + a_i a_i'}{a_i a_i' - a_i'^2} \right) \geq 0, \]
(3.2)

which clearly indicates that the proposed class of estimators \( d_i \) is more efficient than the proposed sub class of estimators \( d_{i(i)} \).

From (3.1) and (3.2) we have the following inequality:
\[
\min \text{MSR}(d_{i(i)}) \leq \text{MSR} \left( \hat{Y}_1 \right) \leq \min \text{MSR}(d_i), \]
(3.3)

Combining the inequalities (3.3) and (3.4) we have the inequalities
\[
\min \text{MSR}(d_i) \leq \min \text{MSR}(d_{i(i)}) \leq \text{MSR} \left( \hat{Y}_1 \right) \leq \min \text{MSR}(d_i), \]
(3.5)

which follows that the proposed class of estimators \( d_i \) is more efficient than the estimators \( \hat{Y}_1 \) and \( d_{i(i)} \). The minimum \( \text{MSE} \) of Ray and Singh (1981) estimator \( \hat{Y}_{R_0} \) in (1.34) can be rewritten as
\[
\min \text{MSR} \left( \hat{Y}_{R_0} \right) = Y^2 \left[ 1 + a_i - 2a_i - C_2 C(C + 2) \right], \]
(3.6)

From (2.26) and (3.6) we have that
\[
\min \text{MSR}(d_i) \leq \min \text{MSR} \left( \hat{Y}_{R_0} \right), \]
if
\[
\left[ a_i - 2a_i - C_2^2 C(C + 2) + \frac{a_i a_i^2 - 2a_i a_i a_i' + a_i a_i'}{a_i a_i' - a_i'^2} \right] \geq 0. \]
(3.7)

Thus the proposed class of estimators \( d_i \) is more efficient than the Ray and Singh (1981) estimators \( \hat{Y}_{R_0} \) as long as the condition (3.7) is satisfied.

4. **Empirical Study**

In this section, we consider the simple random sampling without replacement (SRSWOR) design. We denote
\[
C_i^2 = \frac{S_i^2}{\bar{Y}_i^2}, \quad F_i^2 = \frac{1}{N} \sum_{j=1}^N y_{ij}, \quad C_i^2 = \left( \frac{N-n}{nN} \right) C_i^2 \quad \text{and} \quad S_i^2 = \frac{1}{N-1} \sum_{j=1}^N (y_{ij} - F_i)^2, \quad i = 0, 1, 2.
\]

To observe the relative performance of different estimators of the population total \( Y \) discussed here, we consider a natural population data.
Population: [Source: Singh (1969)]
y = Numbers of female employed,
X₁ = Number of females in service,
X₂ = Number of educated females,
Y = 7.46, C₀ = 0.710, C₀² = 0.504, ρ₀₁ = 0.773, X₁ = 5.31,
C₁ = 0.757, C₁² = 0.573, ρ₀₂ = -0.207, X₂ = 179.00,
C₂ = 0.251, C₂² = 0.063, ρ₂ = -0.003, N = 61.

For illustration, we have taken n = 6. The percent relative efficiencies (PREs) of different estimators of the population total Y with respect to the usual unbiased estimator \( \hat{Y} \) are given in Table 4.1. 

Table 4.1 PREs of various estimators of Y with respect to \( \hat{Y} \)

<table>
<thead>
<tr>
<th>Estimators</th>
<th>( PR(\bullet,\hat{Y}) )</th>
<th>Estimators</th>
<th>( PR(\bullet,\hat{Y}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{Y} )</td>
<td>100.00</td>
<td>( \hat{Y}_j ), j = a, KC, P(1), P(2), P(3), Sand α</td>
<td>278.09</td>
</tr>
<tr>
<td>( \hat{Y}_R )</td>
<td>205.31</td>
<td>( d_{1(i)}^* )</td>
<td>292.76</td>
</tr>
<tr>
<td>( \hat{Y}_1 )</td>
<td>214.06</td>
<td>( d_{1(i)} )</td>
<td>314.97</td>
</tr>
<tr>
<td>( \hat{Y}_{Rθ} )</td>
<td>224.74</td>
<td>( d_i )</td>
<td>422.92</td>
</tr>
<tr>
<td>( d_{1(i)}^* )</td>
<td>226.00</td>
<td>( d ) with ( α_{\text{opt}} = 0.7250 ) ( α_{2\text{opt}} = -0.5821 )</td>
<td>8291.11</td>
</tr>
<tr>
<td>( \hat{Y}_1 )</td>
<td>278.08</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is observed from Table 4.1 that there is substantial gain in efficiency by using the proposed class of estimators \( d_i \) over the usual unbiased estimator \( \hat{Y} \), ratio estimator \( \hat{Y}_R \), ratio-cum-product estimator \( \hat{Y}_1 \) due to Singh (1967), Ray and Singh (1981) estimator \( \hat{Y}_{Rθ} \), Abu-Dayyeh et al. (2003) estimator \( \hat{Y}_a \), Kadilar and Cingi (2005) estimator \( \hat{Y}_{KC} \), Perri (2007) estimator, \( \hat{Y}_{P(i)} \), i = 1, 2, 3; Singh et al. (2009) estimator \( \hat{Y}_S \), regression estimator \( \hat{Y}_l \), the difference estimator \( \hat{Y}_α \) and the estimators \( d_{1(i)}^*, d_{1(i)}^*, d_{l(i)} \). However, the proposed class of estimators ‘ \( d \)’ at optimum values \( (α_{\text{opt}} = 0.7250, α_{2\text{opt}} = -0.5821) \) of \( (α_1, α_2) \) beats all the estimators discussed here.

Thus our recommendations are

(i) to use the proposed class of estimators \( d_i \);

(ii) to use the suggested class of estimators \( d \) if the optimum values \( (α_{\text{opt}}, α_{2\text{opt}}) \) of \( (α_1, α_2) \) are known; and

(iii) to use the suggested class of estimators \( d \) if the values of \( (α_1, α_2) \) are near to optimum values \( (α_{\text{opt}}, α_{2\text{opt}}) \) available; in practice.
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