An Approximated Optimal Stabilization for Solutions of Impulsive Parabolic Problems with Fast Oscillating Coefficients

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Abstract

For the problem of optimal stabilization of solutions of a parabolic equation with rapidly oscillating coefficients, when an impulse controlled perturbation of the system takes place at a fixed moment of time, we justify the form of approximate optimal regulator in which rapidly oscillating coefficients are replaced with their averaged values.

Keywords: Optimal Control; Impulse Perturbation; Homogenization; Inverse Relationship

1. Introduction

An important problem of the optimal control theory is to obtain the optimal regulator, i.e. optimal control in the feedback form. In the case of infinite time interval this problem is called optimal stabilization. For a wide class of linear-quadratic infinite-dimensional problems close-loop optimal controller can be found, parameters of which are expressed via eigenvalues and eigenfunctions of the respective differential operator. If the original problem describes a process in micro-inhomogeneous media, its coefficients and the formula of optimal regulator are typically expressed via rapidly oscillating parameters. This makes impossible the practical implementation of these formulas. For such problems our goal was to construct and justify the use of approximate optimal regulator obtained by replacing rapidly oscillating coefficients with their averaged values as in Jikov et al. (1993).

The aforementioned problem was addressed in Kapustyan et al. (2013) for the controls of the form $g(x)u(t)$ under certain assumptions on the input data and in Kapustyan and Rusina (2015) for the...
control of the form \( u(x, t) \). In this paper, we focus on the optimal stabilization of solutions of a parabolic equation with rapidly oscillating coefficients and distributed control \( u(x, t) \), when impulse controlled perturbation of the system take place at a fixed moment of time as in Samoilenko and Perestyuk (1987). The case of finite time interval was investigated in Kapustyan et al. (2014). Using the exact formula of optimal control in the feedback form, we justify the formula of approximate optimal regulator, in which rapidly oscillating coefficients are replaced with their homogenized values and infinite sums are replaced with finite.

2. Statement of the Problem

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( \epsilon \in (0, 1) \) be a small parameter, \( Q = (0, \infty) \times \Omega, \theta \in (0, \infty) \) be a fixed moment of time of an impulse perturbation, \( a \neq -1, b \in \mathbb{R}, c > 0, d > 0 \) are fixed.

A controlled process \( \{y, u, w\} \) is described by the following relations

\[
\begin{aligned}
\frac{\partial y}{\partial t} = A^\epsilon y + u(t, x), \quad (t, x) \in Q, \\
y|_{t=0} = y_0, \\
y|_{t=\theta} = y^\epsilon, \\
y(\theta + 0, x) - y(\theta, x) = ay(\theta, x) + bw(x) \quad \text{for a.e.} \ x \in \Omega,
\end{aligned}
\]

\( J(y, u, w) = \int_Q (y^2(t, x) + cu^2(t, x))dt \, dx + d \int_\Omega w^2(x)dx \to \inf, \)

where \( A^\epsilon = \text{div}(a^\epsilon \nabla), a^\epsilon(x) = a(C^\epsilon), a \) is a measurable symmetric periodic matrix which satisfies the condition of uniform ellipticity \( \exists \nu_1 > 0, \nu_2 > 0 \ \forall \eta \in \mathbb{R}^n \)

\[
\nu_1 \sum_{i=1}^n \eta_i^2 \leq \sum_{i,j=1}^n a_{ij}(x)\eta_i\eta_j \leq \nu_2 \sum_{i=1}^n \eta_i^2.
\]

Let \( \{X^\epsilon_i\}, \{\lambda^\epsilon_i\} \) be solutions of the following spectral problem

\[
\begin{aligned}
A^\epsilon X^\epsilon_i = -\lambda^\epsilon_i X^\epsilon_i, \\
X^\epsilon_i|_{\partial \Omega} = 0.
\end{aligned}
\]

\( \{X^\epsilon_i\} \subset H^1_0(\Omega) \) is an orthonormal basis in \( L^2(\Omega) \), \( 0 < \lambda^\epsilon_1 \leq \lambda^\epsilon_2 \leq \cdots, \lambda^\epsilon_i \to \infty, i \to \infty. \)

Let \( \| \cdot \| \) and \( (\cdot, \cdot) \) denote the norm and the scalar product in the space \( L^2(\Omega) \) respectively.

We follow Lions (1971) in assuming that the optimal control problem (1)-(3) for every \( \epsilon > 0 \), \( y^\epsilon_0 \in L^2(\Omega) \) has the unique solution \( \{y^\epsilon, u^\epsilon, w^\epsilon\} \) in \( W_0^\epsilon \times L^2(Q) \times L^2(\Omega) \), where \( W_0 \) is a class of functions \( y \in L^2(0, +\infty; H^1_0(\Omega)) \), whose restrictions on \( (0, \theta) \) and \( (\theta, +\infty) \) have generalized derivatives with respect to \( t \) from classes \( L^2(0, \theta; H^{-1}(\Omega)) \) and \( L^2(\theta, +\infty; H^{-1}(\Omega)) \) respectively. In particular, every function of class \( W_0 \) is continuous on \( [0, +\infty) \setminus \{\theta\} \) with respect to the norm of \( L^2(\Omega) \). We will assume that it is left continuous at \( t = \theta \).

The open-loop control \( \{u^\epsilon, w^\epsilon\} \) can be derived using Pontryagin’s maximum principle (see Samoilenko and Perestyuk (1987)) and in the case of unbounded control the optimal regulator
\{u^\varepsilon[t, x, y^\varepsilon(t, x)], \ w^\varepsilon[x, y^\varepsilon(\theta, x)], \ w^\varepsilon[0, x, y(\theta, x)]\}, which is expressed in terms of infinite series of \{X_i^\varepsilon\}, \{\lambda_i^\varepsilon\}, can be found.

The main aim of this paper is to justify that that the formula \{u_0^\varepsilon[t, x, y(t, x)], w_0^\varepsilon[0, x, y(\theta, x)]\}, in which all infinite sums are replaced with finite and parameters \{X_i^\varepsilon\}, \{\lambda_i^\varepsilon\}, are replaced with their homogenized \{X_i^0\}, \{\lambda_i^0\}, defines an approximate optimal control in feedback form of the problem (1)-(3), i.e. \forall \eta > 0

\[ |J(y^\varepsilon, u^\varepsilon, w^\varepsilon) - J(y_N^\varepsilon, u_N^\varepsilon[0], w_N^\varepsilon[0])] < \eta \]

for large enough \(N \geq 1\) and small enough \(\varepsilon > 0\), where \(y_N^\varepsilon\) is the solution of problem (1)-(3) with the control \(u_N^\varepsilon, w_N^\varepsilon\).

3. Construction of the Optimal Regulator

Let us find a solution of (1)-(3) in the form

\[ y^\varepsilon(t, x) = \sum_{i=1}^{\infty} y_i^\varepsilon(t)X_i^\varepsilon(x), \quad u^\varepsilon(t, x) = \sum_{i=1}^{\infty} u_i^\varepsilon(t)X_i^\varepsilon(x), \quad w^\varepsilon(x) = \sum_{i=1}^{\infty} w_i^\varepsilon X_i^\varepsilon(x), \]

where \(\{X_i^\varepsilon\}\) are solutions of the spectral problem (5).

We obtain the countable set of one-dimensional impulsive optimal stabilization problems

\[
\begin{align*}
\frac{d}{dt}y_i^\varepsilon(t) &= -\lambda_i^\varepsilon y_i^\varepsilon + u_i^\varepsilon(t), \\
y_i^\varepsilon(0) &= y_i^{0\varepsilon}, \\
y_i^\varepsilon(\theta + 0) - y_i^\varepsilon(\theta) &= ay_i^\varepsilon(\theta) + bw_i^\varepsilon, \\
J_i(y_i^\varepsilon, u_i^\varepsilon, w_i^\varepsilon) &= \int_0^\infty ((y_i^\varepsilon(t))^2 + c(u_i^\varepsilon(t))^2)dt + d(w_i^\varepsilon)^2 \to \inf,
\end{align*}
\]

where \(y_i^{0\varepsilon} = (y_i^0, X_i^\varepsilon)\).

First, let us consider problem (6) on a finite interval \((0, T), T > \theta\)

\[
\begin{align*}
\frac{d}{dt}y_i^\varepsilon(t) &= -\lambda_i^\varepsilon y_i^\varepsilon + u_i^\varepsilon(t), \\
y_i^\varepsilon(0) &= y_i^{0\varepsilon}, \\
y_i^\varepsilon(\theta + 0) - y_i^\varepsilon(\theta) &= ay_i^\varepsilon(\theta) + bw_i^\varepsilon, \\
J_i^T(y_i^\varepsilon, u_i^T, w_i^T) &= \int_0^T ((y_i^\varepsilon(t))^2 + c(u_i^\varepsilon(t))^2)dt + d(w_i^\varepsilon)^2 \to \inf,
\end{align*}
\]

From Pontryagin’s maximum principle applied to (7) (see Samoilenko and Perestyuk (1987)), there exists \(\psi_i(t)\) such that optimal process \(z^T = \{y_i^T, u_i^T, w_i^T\}\) of the problem (7) which is characterized by the following system
\[
\begin{align*}
\frac{dy^x_i}{dt} &= -\lambda^x_i y^x_i + \psi_i \frac{e^t}{2c}, \quad t \neq \theta, \\
\frac{dy^\psi_i}{dt} &= 2y^x_i + \lambda^\psi_i \psi_i, \quad t \neq \theta, \\
y^x_i(0) &= y^0_i, \quad \psi_i(T) = 0, \\
u_i(t) &= \frac{\psi_i(t)}{2c}, \\
y^x_i(\theta + 0) &= (a + 1)y^x_i(\theta) + \frac{b^2}{2d(a + 1)}\psi_i(\theta), \\
\psi_i(\theta + 0) &= \frac{1}{a + 1}\psi_i(\theta), \\
w_i &= \frac{b}{2d}\psi_i(\theta + 0).
\end{align*}
\]

Thus,
\[
\begin{align*}
y^x_i(\theta + t) &= \tilde{C}_k^{i,e}(T) \frac{1}{2c} e^{\sqrt{\left(\lambda^x_i\right)^2 + \frac{1}{c}^2} t} + \tilde{C}_e^{i,\theta}(T) \frac{1}{2c} e^{-\sqrt{\left(\lambda^x_i\right)^2 + \frac{1}{c}^2} t}, \\
u_i(\theta + t) &= \frac{c_{i,e}(T)(\lambda^2 + \sqrt{\lambda^2 + \frac{1}{c}^2}) + c_{e}(T)(\lambda^2 - \sqrt{\lambda^2 + \frac{1}{c}^2})}{2c} e^{\sqrt{\lambda^2 + \frac{1}{c}^2} t}, \\
w_i(\theta) &= \frac{bc}{(a + 1)d} u_i^T(\theta),
\end{align*}
\]

where
\[
\tilde{C}_k^{i,e}(T) = \begin{cases} c_k^{i,e}(T), & t \in [0, \theta], \\
\tilde{C}_k^{i,e}(T), & t \in (\theta, T], \\
k = 1, 2,
\end{cases}
\]

and parameters $C_1^{i,e}(T), C_2^{i,e}(T), \tilde{C}_1^{i,e}(T), \tilde{C}_2^{i,e}(T)$ are determined by the following algebraic system
\[
\begin{pmatrix}
\frac{1}{2c} \\
0 \\
-\frac{(a + 1)^2 + b^2 c(\lambda^2 + k^2)}{2d(a + 1)} e^{k^2 \theta} \\
-\frac{1}{a + 1} (\lambda^2 + k^2) e^{-k^2 \theta}
\end{pmatrix}
\begin{pmatrix}
C_1^{i,e} \\
C_2^{i,e} \\
\tilde{C}_1^{i,e} \\
\tilde{C}_2^{i,e}
\end{pmatrix}
= 
\begin{pmatrix}
y^x_i(0) \\
0 \\
0 \\
0
\end{pmatrix}.
\]

When $T \to \infty$ we obtain
\[
\begin{align*}
C_1^{i,e} &= \lim_{T \to \infty} C_1^{i,e}(T) = -2 \frac{c y^0_i}{\mu_1} e^{k^2 \theta} \left(\frac{e^{k^2 \theta} - \mu^e_1}{k^2} - \mu_1^e \right)^{-1}, \\
C_2^{i,e} &= \lim_{T \to \infty} C_2^{i,e}(T) = 2c y^0_i e^{2k^2 \theta} \left(\frac{e^{2k^2 \theta} - \mu^e_1}{k^2} - \mu_1^e \right)^{-1}, \\
\tilde{C}_1^{i,e} &= \lim_{T \to \infty} \tilde{C}_1^{i,e}(T) = 0,
\end{align*}
\]
\[
\tilde{C}_2^{ie} = \lim_{T \to \infty} \tilde{C}_2^{ie} (T) = 4c^2 \gamma_i^{0e} k_i e^{2k_i \theta} (\lambda_i^{e} + k_i) \sigma_i^{e} \left((a + 1)(\tau_i^{e} e^{2k_i \theta} - \mu_i^{e})\right)^{-1},
\]

where
\[
\begin{align*}
\tau_i^{e} &= (a + 1)^2 d + b^2 c(\lambda_i^{e} + k_i) + dc(\lambda_i^{e} + k_i)^2, \\
\mu_i^{e} &= (a + 1)^2 d + b^2 c(\lambda_i^{e} - k_i) - d, \\
o_i^{e} &= (a + 1)^2 d + b^2 c(\lambda_i^{e} + k_i).
\end{align*}
\]

Let us consider
\[
\begin{align*}
y_i^{e}(t) &= c_1^{ie} \frac{1}{2c} e^{\left(\lambda_i^{e} \tau_i^{e} + \frac{1}{2}\right)} + \tilde{C}_2^{ie} \frac{1}{2c} e^{-\left(\lambda_i^{e} \tau_i^{e} + \frac{1}{2}\right)}, \\
u_i^{e}(t) &= \frac{c_1^{ie}(\lambda_i^{e} + \frac{1}{2}) e^{\left(\lambda_i^{e} \tau_i^{e} + \frac{1}{2}\right)} + \tilde{C}_2^{ie}(\lambda_i^{e} - \frac{1}{2}) e^{-\left(\lambda_i^{e} \tau_i^{e} + \frac{1}{2}\right)}}{2c}, \\
w_i^{e} &= \frac{bc}{(a+1)d} u_i^{e}(\theta),
\end{align*}
\]

where
\[
\begin{align*}
\tilde{C}_k^{ie} = \begin{cases} C_k^{ie}, & t \in [0, \theta], \\ \tilde{C}_k^{ie}, & t > \theta, \end{cases},
\end{align*}
\]

where constants \(C_1^{ie}, C_2^{ie}, \tilde{C}_1^{ie}, \tilde{C}_2^{ie}\) are from (12). Then process (13) is optimal in (6). From (9)-(12) we have \(\{y_i^{e}, u_i^{e}\} \in L^2(0, +\infty)\) and
\[
J^I(y_i^{eT}(t) - y_i^{e}(t))^2 dt + \int_0^T (u_i^{e}(t) - u_i^{eT}(t))^2 dt + (w_i^{e} - w_i^{eT})^2 \to 0, \quad T \to \infty.
\]

Then for any feasible process \(\{\tilde{y}_i^{e}, \tilde{u}_i^{e}, \tilde{w}_i^{e}\}\) of the problem (6) and for all \(T > \theta\) we have
\[
J^I(\tilde{y}_i^{eT}, \tilde{u}_i^{eT}, \tilde{w}_i^{eT}) = J^I(\tilde{y}_i^{eT}, \tilde{u}_i^{eT}, \tilde{w}_i^{eT}) + \int_0^\infty ((\tilde{y}_i^{e}(t))^2 + c(\tilde{u}_i^{e}(t))^2) dt \geq J^I(y_i^{eT}, u_i^{eT}, w_i^{eT}).
\]

But with (14)
\[
J^I(y_i^{eT}, u_i^{eT}, w_i^{eT}) \to J^I(y_i^{e}, u_i^{e}, w_i^{e}), \quad T \to \infty,
\]

and it proves optimality of (13).

Note that in case without an impulse perturbation \((a = b = 0)\) the formula for optimal control of (6) is known. It is of the form
\[
\forall \ t \geq 0 \ \ u_i^{e}(t) = y_i^{0e} \left(\lambda_i^{e} - \sqrt{(\lambda_i^{e})^2 + \frac{1}{c}}\right) e^{-\sqrt{(\lambda_i^{e})^2 + \frac{1}{c}}}.
\]

By substitution \(\{u_i^{e}, w_i^{e}\}\) into (6) and removing \(y_i^{0e}\), we obtain a formula of the optimal regulator of problem (6).
\[ u^\varepsilon [t, y^\varepsilon (t)] = \beta^\varepsilon (t)y^\varepsilon (t), \quad \text{(16)} \]

\[ w^\varepsilon [y^\varepsilon (\theta)] = \frac{bc}{d(a+1)} \beta^\varepsilon (\theta)y^\varepsilon (\theta), \quad \text{(17)} \]

where

\[
\beta^\varepsilon (t) = \begin{cases} 
\frac{(\lambda^\varepsilon_i - k^\varepsilon_i)e^{(\lambda^\varepsilon_i - k^\varepsilon_i)\theta} - (\lambda^\varepsilon_i + k^\varepsilon_i)e^{(\lambda^\varepsilon_i + k^\varepsilon_i)\theta} \mu^\varepsilon_i}{e^{(\lambda^\varepsilon_i + k^\varepsilon_i)\theta} \tau^\varepsilon_i - e^{(\lambda^\varepsilon_i - k^\varepsilon_i)\theta} \mu^\varepsilon_i}, & t \in [0, \theta], \\
\frac{2ck^\varepsilon_i(\lambda^\varepsilon_i + k^\varepsilon_i)(\lambda^\varepsilon_i - k^\varepsilon_i)\epsilon^\varepsilon_i e^{(\lambda^\varepsilon_i - k^\varepsilon_i)\tau_i} e^{2k^\varepsilon_i\theta}}{e^{(\lambda^\varepsilon_i + k^\varepsilon_i)\theta} (a + 1)(\tau_i - \mu_i) + (e^{(\lambda^\varepsilon_i + k^\varepsilon_i)\tau_i} - e^{(\lambda^\varepsilon_i - k^\varepsilon_i)\theta})e^{2k^\varepsilon_i\theta}2ck^\varepsilon_i(\lambda^\varepsilon_i + k^\varepsilon_i)\epsilon^\varepsilon_i}, & t > \theta,
\end{cases}
\]

where \( k^\varepsilon_i = \sqrt{(\lambda^\varepsilon_i)^2 + \frac{1}{\epsilon}} \)

We note that from the formula above it follows that the functions \( \beta^\varepsilon \) are uniformly bounded on \([0, \infty)\), i.e.

\[ \exists \beta > 0 \forall i \geq 1 \forall \epsilon \in (0,1) \sup_{t \in [0, +\infty)} |\beta^\varepsilon_i (t)| \leq \beta. \quad \text{(18)} \]

Denote

\[ y^\varepsilon_i = \frac{bc}{d(a+1)} \beta^\varepsilon_i (\theta). \]

Thus the optimal control in the feedback form for the problem (1)-(3) has the form

\[ u^\varepsilon [t, x, y^\varepsilon (t, x)] = \sum_{i=1}^{\infty} \beta^\varepsilon_i (t)(y^\varepsilon (t), X^\varepsilon_i (x)), \quad \text{(19)} \]

\[ w^\varepsilon [x, y^\varepsilon (\theta, x)] = \sum_{i=1}^{\infty} y^\varepsilon_i (y^\varepsilon (\theta), X^\varepsilon_i (x)). \quad \text{(20)} \]

### 4. Justification of the Approximate Optimal Regulator

Let \( a^0 \) be a constant homogenized matrix for \( a \left( \frac{x}{\varepsilon} \right) \) (see Jikov et al. (1993)), \( A^0 = \text{div}(a^0 \nabla) \), \( \{X^0_i\}, \{\lambda^0_i\} \) are solutions of following spectrum problem

\[
\begin{cases}
A^0 X^0_i = -\lambda^0_i X^0_i, \\
X^0_i|_{\partial \Omega} = 0,
\end{cases}
\]

and let the spectrum of \( A^0 \) be simple, i.e.,

\[ 0 < \lambda^0_1 < \lambda^0_2 < \cdots < \lambda^0_k < \cdots, \lambda^0_i \to \infty, i \to \infty. \quad \text{(21)} \]

With condition (21)

...
The main result of this paper is the following theorem.

Let us define an approximate (parametric) regulator as

$$
\beta^0_i(t) = \begin{cases} 
\frac{(\lambda^0_i - k^0_i) e^{(\lambda^0_i - k^0_i)t} e^{2k^0_i\theta} \tau^0_i - (\lambda^0_i + k^0_i) e^{(\lambda^0_i + k^0_i)t} \mu^0_i}{e^{(\lambda^0_i - k^0_i)t} e^{2k^0_i\theta} \tau^0_i - e^{(\lambda^0_i + k^0_i)t} \mu^0_i}, & t \in [0, \theta], \\
2ck^0_i(\lambda^0_i + k^0_i)(\lambda^0_i - k^0_i)\theta \nu_i^0 e^{(\lambda^0_i - k^0_i)t} e^{2k^0_i\theta} (e^{(\lambda^0_i - k^0_i)t} e^{2k^0_i\theta} - e^{(\lambda^0_i + k^0_i)t} e^{2k^0_i\theta}) 2ck^0_i(\lambda^0_i + k^0_i) \theta, & t > \theta,
\end{cases}
$$

where $k^0_i = \sqrt{(\lambda^0_i)^2 + \frac{1}{c}}$.

Let us define an approximate (parametric) regulator as

$$
u^0_N(t, x, y_N^0(t, x)) = \sum_{i=1}^N \beta^0_i(t)(y^0_i(t), X^0_i(x)),
$$

$$
w^0_N(x, y_N^0(\theta, x)) = \sum_{i=1}^N y^0_i(y^0_i(\theta), X^0_i(x)),
$$

where $y^0_N(t, x)$ is a solution of (1) with the control (24)-(25).

The main result of this paper is the following theorem.

**Theorem 1.** Suppose that the assumptions (4), (21) - (23) are satisfied. Then formulas (24)-(25) determines the approximate optimal regulator of the problem (1)-(3), i.e. $\forall \eta > 0 \ \forall 0 < r < T \ \exists \bar{\epsilon} \in (0,1) \ \exists N \geq 1$ such that $\forall \epsilon \in (0,\bar{\epsilon}) \ \forall N \geq N$

$$
\|u^\epsilon[y^\epsilon] - u^0_N[y_N^0]\|_{L^2(\Omega)} < \eta,
$$

$$
\|w^\epsilon[y^\epsilon(\theta, x)] - w^0_N[y_N^0(\theta, x)]\|_{L^2(\Omega)} < \eta,
$$

$$
\max_{t \in [r, +\infty]} \|y^\epsilon(t) - y^0_N(\theta)\| < \eta,
$$

$$
|J(y^\epsilon, u^\epsilon, w^\epsilon[y^\epsilon(\theta)]) - J(y^0_N, u^0_N[y_N^0], w^0_N[y_N^0(\theta)])| < \eta.
$$

where $(y^\epsilon, u^\epsilon, w^\epsilon)$ is the optimal process of the problem (1)-(3), $y^\epsilon_N$ is the solution of (1)-(2) with the control (24)-(25).

**Proof.** Consider
\[
\frac{\partial z}{\partial t} = A^\varepsilon z + u^0[t, x, z], \\
z|_{t=0} = 0, \\
z|_{t=0} = y_0^\varepsilon,
\]

(30)

\[z(\theta + 0, x) - z(\theta, x) = az(\theta, x) + bw^0(x),\]

(31)

where

\[u^0[t, x, z] = \sum_{i=1}^{\infty} \beta_i^0(t)(z(t), X_i^0)(x),\]

\[w^0[x, z] = \sum_{i=1}^{\infty} \gamma_i^0(z(t), X_i^0)(x).\]

Note that

\[\|u^0[t, z]\| \leq \beta \|z\|, \quad \|u^0[t, z_1] - u^0[t, z_2]\| \leq \beta \|z_1 - z_2\|,\]

\[\|w^0[x, z_1] - w^0[x, z_2]\| \leq \frac{bc}{d(a+1)} \beta \|z_1 - z_2\|,\]

(32)

The impulsive problem (30)-(31) has unique solution \(z^\varepsilon(t, x)\) in \(W_0(0, T)\), \(\forall T > \theta\), which is defined on \([0, +\infty)\).

For a.e. \(t \in (0, +\infty)\) for \(z^\varepsilon\) the following estimate holds

\[\frac{1}{2} \frac{d}{dt} \|z^\varepsilon(t)\|^2 + v_1 \|z^\varepsilon(t)\|^2_{H_0^1} \leq \sum_{i=1}^{\infty} \beta_i^0(z^\varepsilon(t), X_i^0)^2.\]

(33)

Since \(|\beta_i^0(t)| \leq \beta\) and \(\beta_i^0(t) < 0\), we have:

\[\frac{1}{2} \frac{d}{dt} \|z^\varepsilon(t)\|^2 + v_1 \|z^\varepsilon(t)\|^2_{H_0^1} \leq 0.\]

(34)

Thus,

\[\|z^\varepsilon(t)\| \leq \|y_0^\varepsilon\| e^{-v_1 \lambda t},\]

(35)

where a constant \(\lambda > 0\) is from the Poincare inequality. Denote

\[J_T(y, u) = \int_0^T \int_\Omega (y^2(t, x) + u^2(t, x)) dt dx + \int_\Omega w^2(x) dx,\]

then the following convergence is proved in Kapustyan et al. (2014)

\[J_T(z^\varepsilon, u^0[z^\varepsilon]) \to J_T(z, u^0[z]), \varepsilon \to 0.\]

(36)

Let us prove this convergence on an infinite interval. Since
Let us compare solutions $\psi_N^\varepsilon$ and $z^\varepsilon$. For $\psi_N^\varepsilon = \psi_N^0 - z^\varepsilon$ we have the following problem

\[
\begin{align*}
\frac{\partial \psi_N^\varepsilon}{\partial t} &= A^\varepsilon \psi_N^\varepsilon + \sum_{i=1}^{N} \beta_i^0(t)(\psi_N^\varepsilon(t),X_i^0)(x) + f_i^\varepsilon(t,x), \\
\psi_N^\varepsilon|_{t=0} &= 0, \\
\psi_N^\varepsilon|_{t=0} &= 0,
\end{align*}
\]

(42)

\[
\psi_N^\varepsilon(\theta + 0,x) - \psi_N^\varepsilon(\theta,x) = a\psi_N^\varepsilon(\theta,x) + b\sum_{i=1}^{N} \gamma_i^0(\psi_N^\varepsilon(\theta),X_i^0)(x) + g_i^\varepsilon(x),
\]

(43)

where

\[
f_i^\varepsilon(t,x) = - \sum_{i=N+1}^{\infty} \beta_i^0(z^\varepsilon(t),X_i^0)(x),
\]

\[
g_i^\varepsilon(x) = - \sum_{i=N+1}^{\infty} \gamma_i^0(z^\varepsilon(\theta),X_i^0)(x).
\]

It is known from Kapustyan et al. (2014) that $\forall \varepsilon > 0 \exists N_1 \geq 1, \exists \varepsilon_1 \in (0,1) \forall N \geq N_1 \forall \varepsilon \in (0,\varepsilon_1)$

\[
\sup_{t \in [0,T]} \|\psi_N^\varepsilon(t)\|^2 + \int_0^T \|u_0^\varepsilon[\psi_N^\varepsilon] - u^0[z^\varepsilon]\|^2 dt < \eta,
\]

(44)

\[
|J_T(\psi_N^\varepsilon, u_0^\varepsilon[\psi_N^\varepsilon], w_0^\varepsilon[\psi_N^\varepsilon(\theta)]) - J_T(z^\varepsilon, u^0[z^\varepsilon], w^0[z^\varepsilon(\theta)])| < \eta,
\]

For all $t \geq 0$
Let us show that

\[ \| f_N^\epsilon(t) \|^2 = \sum_{i=N+1}^{\infty} (\beta^\epsilon_i (x^\epsilon(t), X^0_i))^2 \leq 2\beta^0_{N+1} \| z^\epsilon(t) \|^2 \leq (\text{with (35)}) \leq m^0_{N+1} \to 0, \quad N \to \infty. \]

Thus,

\[ \frac{1}{2} \frac{d}{dt} \| v_N^\epsilon(t) \|^2 + v_1^\epsilon \| v_N^\epsilon(t) \|^2 \leq \| f_N^\epsilon(t) \| v_N^\epsilon(t). \]

It follows that \( \forall t \geq 0 \)

\[ \| v_N^\epsilon(t) \|^2 \leq \| v_N(0) \|^2 e^{-\lambda v_1 t} + e^{-\lambda v_1 t} \int_0^t \frac{m^2_{N+1}}{\lambda^2 v_1} e^{\lambda v_1 s} ds \leq \frac{m^2_{N+1}}{\lambda^2 v_1} e^{-\lambda v_1 t}. \]

Then,

\[ \lambda v_1 \int_{2T}^{\infty} \| v_N^\epsilon(t) \|^2 dt \leq \| v_N(2T) \|^2 + \beta^0_{N+1} \int_{2T}^{\infty} \| z^\epsilon(t) \|^2 dt \leq \frac{m^2_{N+1}}{\lambda^2 v_1} + \beta^0_{N+1} e^{-2\lambda v_1 T}. \]

Combining the above estimate with (44) we have \( \forall \eta > 0 \exists \epsilon \exists N \forall \epsilon \in (0, \tilde{\epsilon}) \forall N \geq \tilde{N} \)

\[ \| v_N^\epsilon \|_{L^2(Q)} < \eta. \]

Since

\[ \int_{2T}^{\infty} \| u_N^0 [y^\epsilon] - u^0 [z^\epsilon] \|^2 dt = \int_{2T}^{\infty} \| \sum_{i=1}^{\infty} \beta_i^0 (v_N^\epsilon(t), X^0_i) X^0_i + f^\epsilon \|^2 dt \leq \beta^0_{N+1} \int_{2T}^{\infty} \| v_N^\epsilon(t) \|^2 dt + \beta^0_{N+1} \int_{2T}^{\infty} \| z^\epsilon(t) \|^2 dt \leq C \beta^0_{N+1}, \]

we obtain the required estimates on \([0, +\infty)\).

It remains to prove that \( \{ y^\epsilon, u^\epsilon [y^\epsilon] \} \) converges to \( \{ z, u^0 [z] \} \) in the sense of (26)-(29). From Kapustyan et al. (2014)

\[ y^\epsilon \to y \text{ in } C([\delta, T]; L^2(\Omega)), \]

\[ u^\epsilon [y^\epsilon] \to u^0 [y] \text{ in } L^2(0, T; L^2(\Omega)). \]

\[ f_T (y^\epsilon, u^\epsilon [y^\epsilon], w^\epsilon [y^\epsilon(\theta)]) \to f_T (y, u^0 [y], u^0 [y(\theta)]), \]

where \( y = z \) is a solution of (30) with \( \epsilon = 0 \).

Let us show that \( u^\epsilon [y^\epsilon] \to u^0 [y] \) in \( L^2(Q) \). For this purpose, we use the fact that the process \( \{ y^\epsilon, u^\epsilon \} \) is optimal. By Bellman's principle of optimality, we obtain

\[ \int_{2T}^{\infty} \| y^\epsilon(t) \|^2 + \| u^\epsilon(t) \|^2 dt \leq \int_{2T}^{\infty} \| y^\epsilon(t) \|^2 dt, \]

where \( y^\epsilon \) is a solution of (1) on \([2T, +\infty)\) with the control \( u \equiv 0 \) and initial condition \( y^\epsilon(2T) = y^\epsilon(2T) \).

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Then
\[ \int_{2T}^{\infty} \|	ilde{y}(t)\|^2 dt \leq \sum_{i=1}^{\infty} \frac{1}{2A_i} (\gamma_i^\varepsilon(2T))^2. \]  (50)

Following Kapustyan et al. (2013) we have
\[ \exists \varepsilon > 0 \ \forall i \geq 1 \ \forall \varepsilon \in (0,1) \frac{1}{\lambda_i^\varepsilon} \leq \frac{\varepsilon}{\lambda_i^\varepsilon}. \]

Then with (30) \( \forall \eta > 0 \ \exists i_0 \geq 1 \ \forall \varepsilon \in (0,1) \)
\[ \sum_{i=i_0+1}^{\infty} \frac{1}{2A_i} (\gamma_i^\varepsilon(2T))^2 \leq \frac{\varepsilon}{2A_{i_0+1}} \|y^\varepsilon(2T)\|^2 < \frac{\varepsilon}{\lambda_{i_0+1}^\varepsilon} < \frac{\eta}{2}. \]  (51)

Thus for \( i \in \overline{1,i_0} \) on \([T, 2T]\)
\[ y_i^\varepsilon(2T) \leq y_i^\varepsilon(T) e^{-\lambda_i^{\varepsilon}t} \leq y_i^\varepsilon(T) e^{-\lambda_i^{\varepsilon}t}. \]

It follows that
\[ \sum_{i=1}^{i_0} \frac{1}{2A_i} (\gamma_i^\varepsilon(2T))^2 \leq \frac{\varepsilon}{2A_i} e^{-2\nu_i^\varepsilon T} \|y^\varepsilon(T)\|^2. \]  (52)

With (51), (52) we derive
\[ y^\varepsilon \to y \ \text{in} \ \mathbb{L}^2(Q). \]

In the same manner, when \( \varepsilon = 0 \), we can see that
\[ u^\varepsilon[y^\varepsilon] \to u^0[y] \ \text{in} \ \mathbb{L}^2(Q) \]

and the proof is complete.

5. Conclusion

In this paper, we obtained the exact formula of optimal regulator for parabolic equation with rapidly oscillating coefficients, when impulse controlled perturbation of the system take place at a fixed moment of time. The formula contains rapidly oscillating coefficients and infinite sums. Since the numerical implementation of it is problematic we justified the use of approximate optimal regulator, obtained by replacing rapidly oscillating coefficients with their homogenized values and infinite sums with finite sums.

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