Thermocapillary Migration of a Fluid Sphere in a Circular Tube

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Abstract

The thermocapillary motion of a spherical fluid drop along the axis of a circular tube, whose wall may be either insulated or prescribed with a linear temperature distribution, is investigated theoretically in the steady limit of negligible Marangoni and Reynolds numbers. The proximity of the tube wall causes two effects on the thermocapillary migration velocity of the drop: the temperature gradients along the drop surface are altered, thereby the drop is slowed down or speeded up, and the viscous drag on the drop is enhanced. The general solutions to the energy and momentum equations are constructed in combined circular cylindrical and spherical coordinates, and then the boundary conditions are enforced on the tube wall by the Fourier transform and along the drop surface by a collocation method. The collocation results for the normalized drop velocity, obtained as decreasing functions of the drop-to-tube radius ratio, are in good agreement with the asymptotic formulas derived from a method of reflections. The normalized thermocapillary mobility increases/decreases with an increase in the relative thermal conductivity of the drop for the case of a conducting/insulating tube wall and increases with an increase in the relative viscosity of the drop. The boundary effect on the thermocapillary migration of a drop in a tube, which is relatively weak in comparison with the corresponding effect on its sedimentation, is found to be conspicuously different from that in a slit with two plane walls.

Keywords: Thermocapillary Motion; Spherical Drop; Gas Bubble; Boundary Effect; Circular Tube

1. Introduction

A drop of one fluid suspended in a second, immiscible fluid possessing a temperature gradient will migrate towards the hotter side because of the thermally induced interfacial tension gradient along the drop surface (Marangoni effect). The thermocapillary migration of fluid drops, which was first demonstrated experimentally and analyzed mathematically by Young et al. (1959), plays an

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important role in many practical applications (Subramanian and Balasubramaniam, 2001; Selva et al., 2011). The migration velocity \( U_0 \) of a drop of radius \( a \) in an unbounded fluid of viscosity \( \eta \) with a linearly prescribed temperature distribution \( T_\infty \) is related to the uniform temperature gradient by (Young et al., 1959)

\[
U_0 = M_T \nabla T_\infty,
\]

where the drop's thermocapillary mobility

\[
M_T = \frac{2(-\partial \gamma / \partial T) a}{(2 + k^*)(2 + 3\eta^*) \eta},
\]

provided that the capillary number \( \eta |U_0| / \gamma \) is small for the drop to remain its spherical shape and the convective transports of energy and momentum are negligible. In Eq. (2), \( k^* \) and \( \eta^* \) are the thermal conductivity and viscosity, respectively, of the drop relative to the ambient fluid, \( \gamma \) is the interfacial tension taken to be a linear decreasing function of the temperature \( T \). The limiting case of \( k^* = \eta^* = 0 \) for the system can denote a gas bubble in a liquid.

In many real situations of thermocapillary motion, fluid drops are not isolated (Meyyappan et al., 1983; Anderson, 1985; Keh and Chen, 1990, 1992, 1993; Kasumi et al., 2000; Nas et al., 2006; Yin et al., 2011; Katz et al., 2012; Liao et al., 2014) and it is needed to understand how the proximity of a boundary affects the applicability of Eq. (2) for a fluid drop. The quasi-steady problems of thermocapillary migration of a spherical gas bubble or liquid drop normal to an infinite planar solid or fluid surface were solved by using a representation in spherical bipolar coordinates (Meyyappan et al., 1981; Barton and Subramanian, 1990; Chen and Keh, 1990; Grashchenkov, 2012), a boundary integral technique (Ascoli and Leal, 1990), a method of reflections (Chen and Keh, 1990), and a lubrication approach (Loewenberg and Davis, 1993). These exact numerical solutions and asymptotic analytical results indicate that the migration velocity of the confined drop relative to that of an isolated drop decreases with its approach to the boundary and in general increases with increases in \( k^* \) and \( \eta^* \), as a result of the boundary-drop thermal and hydrodynamic interactions. Experimental data for the thermocapillary migration of a fluid drop perpendicular to a solid plane wall (Barton and Subramanian, 1991) agree well with the theoretical predictions. Similar results have also been obtained for the thermocapillary motions of a fluid sphere normal to two plane walls at an arbitrary position between them (Chang and Keh, 2006) and in an arbitrary direction inside a spherical cavity (Lee and Keh, 2013a, 2013b) by using a boundary collocation method.

Furthermore, the thermocapillary migration of a spherical gas bubble parallel to a plane wall prescribed with a linear temperature profile consistent with the far-field distribution was examined by using spherical bipolar coordinates (Meyyappan and Subramanian, 1987) and the thermocapillary motion of a fluid sphere parallel to one or two plane walls (either prescribed with the linear temperature profile or insulated) at an arbitrary position between them was studied by using both the boundary collocation technique and the method of reflections (Keh et al., 2002). Again, the normalized migration velocity of the confined drop was found to increase as \( \eta^* \) increases and in general to decrease with its approach to the boundary. However, for the case of insulated plane walls, the normalized drop velocity increases as \( k^* \) decreases (because more heat is
conducted through the relatively conductive gap between the drop and insulated wall, creating a greater interfacial tension gradient along the drop); under the situation of large \( \eta^* \) and small \( k^* \), the thermocapillary mobility of the drop can increase with a decrease in the drop-to-wall distance (the wall effect of hydrodynamic resistance on the drop is in competition with that of thermocapillary enhancement) as this distance is small. For the case of plane walls imposed with the far-field temperature distribution, the normalized drop velocity increases as \( k^* \) increases (because the conducting wall allows a larger interfacial tension gradient to develop along the drop surface while more energy is conducted through the relatively conductive drop); under the situation of large \( \eta^* \) and small \( k^* \), the thermocapillary mobility of the drop can increase with a decrease in the drop-to-wall distance. When this distance becomes very small, the drop can even move faster than it would if isolated (the wall effect of thermocapillary enhancement on the drop can override that of hydrodynamic resistance) for both cases.

On the other hand, Chen et al. (1991) solved for the thermocapillary motion of a fluid sphere along the centerline of an insulated circular tube for the drop-to-tube radius ratio up to 0.9 using the boundary collocation technique. The normalized migration velocity of the confined drop was found to decrease monotonically with an increase in the radius ratio and to increase as \( k^* \) decreases, but to decrease as \( \eta^* \) increases. Recently, Mahesri et al. (2014) investigated the effect of interface deformability on the axisymmetric thermocapillary migration of a fluid drop (with a finite capillary number) in an insulated circular tube using the boundary integral method. They obtained that the normalized drop velocity increases as \( \eta^* \) increases [opposite to the result of Chen et al. (1991) but consistent with the outcome of the migration parallel to one or two plane walls (Keh et al., 2002)] or \( k^* \) decreases and decreases monotonically with an increase in the drop-to-tube radius ratio.

In this paper, our purpose is to obtain exact numerical results and asymptotic analytical solutions for the thermocapillary migration of a fluid sphere along the axis of a circular tube. The tube wall may be either insulated or prescribed with the linear far-field temperature distribution. For the case of a drop with a relatively low thermal conductivity undergoing thermocapillary motion near an insulated tube wall or of a drop with a relatively high conductivity near a tube wall with the far-field temperature distribution, the heat conduction around the drop will generate larger temperature gradients along the drop surface relative to those in an unbounded medium. These gradients enhance the thermocapillary migration velocity, which will simultaneously be retarded by the viscous interaction of the migrating drop with the wall. Both the effect of the thermal enhancement and the effect of the hydrodynamic retardation increase as the ratio of drop-to-tube radii increases. A main target of this work is to determine if the effect of the thermal enhancement can be overriding at small drop-wall gap widths [like in the migration parallel to one or two plane walls (Keh et al., 2002)]. To ascertain whether the normalized drop velocity increases or decreases with an increase in \( \eta^* \) is another object.

2. Analysis

Consider the steady thermocapillary migration of a spherical fluid drop of radius \( a \) in an immiscible fluid along the axis of a long circular tube of radius \( b \), as shown in Fig. 1. Here \((\rho, \phi, z)\)
and \((r, \theta, \phi)\) denote the circular cylindrical and spherical coordinate systems, respectively, and the origin of coordinates is chosen at the drop center. A linear temperature field \(T_\infty(z)\) with a constant \(\nabla T_\infty = E_\infty \hat{e}_z\) is imposed in the external fluid far from the drop, where \(\hat{e}_z\) is the unit vector in the \(z\) direction and \(E_\infty\) is taken to be positive. The capillary number is assumed to be small so that the interfacial tension maintains the spherical shape of the drop during the confined thermocapillary migration. The purpose is to obtain the correction to Eq. (2) for the drop mobility due to the presence of the tube wall.

![Geometrical sketch for the thermocapillary motion of a spherical drop along the axis of a circular tube.](image)

**Fig. 1.** Geometrical sketch for the thermocapillary motion of a spherical drop along the axis of a circular tube.

### 2.1 Temperature Distribution

For the heat transfer in thermocapillary motions, the Pecket (Marangoni) number is usually small. Hence, the energy equations governing the temperature distribution are

\[
\nabla^2 T = 0 \quad (r \geq a) \tag{3}
\]

for the external fluid and

\[
\nabla^2 \hat{T} = 0 \quad (r \leq a) \tag{4}
\]

for the fluid inside the drop.

The boundary conditions at the drop surface require that the temperature and the normal component of heat flux be continuous, namely,

\[
r = a: \quad T = \hat{T}, \quad k \frac{\partial T}{\partial r} = k \frac{\partial \hat{T}}{\partial r}, \tag{5a}
\]

where \(k\) and \(\hat{k}\) are the thermal conductivities of the external and internal fluids, respectively. Since the temperature field far away from the drop approaches \(T_\infty\), we can write

\[
|z| \to \infty: \quad T \to T_\infty = T_0 + E_\infty z, \tag{6}
\]

\[
\rho = b: \quad T = T_0 + E_\infty z, \tag{7}
\]

where \(T_0\) is the undisturbed temperature at the drop center. In Eq. (7), the tube wall is prescribed.
with a linear temperature profile consistent with the far-field temperature distribution. For the case of a tube whose wall is insulated, this Dirichlet boundary condition should be replaced by the Neumann condition

\[ \rho = b : \quad \frac{\partial T}{\partial \rho} = 0. \] (8)

The temperature distribution satisfying Eqs. (3), (4), and (6) can be expressed as (Chen et al., 1991)

\[ T = T_0 + E_n z + E_n \int_0^\infty W(\omega) l_0(\omega \rho) \sin(\omega z) d\omega + E_n \sum_{m=1}^\infty R_m r^{-m-1} P_m(\cos \theta), \] (9)

\[ \hat{T} = T_0 + E_n \sum_{m=1}^\infty \hat{R}_m r^m P_m(\cos \theta), \] (10)

where \( l_n \) is the modified Bessel function of the first kind of order \( n \), \( P_m \) is the Legendre polynomial of order \( m \), \( W(\omega) \) is an unknown function of the separation variable \( \omega \), \( R_m \) and \( \hat{R}_m \) are unknown constants, and \( m \) is odd. Substituting Eq. (9) into Eq. (7) or (8) and applying the Fourier sine transform on the variable \( z \) lead to a solution for \( W(\omega) \) in terms of \( R_m \) and Eq. (9) becomes

\[ T = T_0 + E_n z + E_n \sum_{m=1}^\infty R_m \delta_m^{(1)}(r, \theta), \] (11)

where the function \( \delta_m^{(1)}(r, \theta) \) is defined by Eq. (B.1) in Appendix B.

Applying Eq. (5) to Eqs. (10) and (11) yields

\[ \sum_{m=1}^\infty [R_m \delta_m^{(1)}(a, \theta) - \hat{R}_m a^m P_m(\cos \theta)] = -ac \cos \theta, \] (12a)

\[ \sum_{m=1}^\infty [R_m \delta_m^{(2)}(a, \theta) - \hat{R}_m k^m a^{m-1} P_m(\cos \theta)] = -c \cos \theta, \] (12b)

where the definition of the functions \( \delta_m^{(2)}(r, \theta) \) is given by Eq. (B.2), \( k^* = \hat{k}/k \), and \( m \) is odd.

To satisfy the conditions in Eq. (12) exactly along the surface of the drop would require the solution of the entire infinite array of the unknown constants \( R_m \) and \( \hat{R}_m \). However, the collocation technique (Chen et al., 1991; Lee and Keh, 2014) enforces the boundary conditions at a finite number of points on the quarter-circular generating arc of the sphere (from \( \theta = 0 \) to \( \theta = \pi/2 \), because of the symmetry of the system geometry and anti-symmetry in the temperature field) and truncates the infinite series in Eqs. (10) and (11) into finite ones. If the spherical boundary is approximated by satisfying the conditions (5a) and (5b) at \( M \) discrete points on the generating arc, the infinite series in Eqs. (10) and (11) are truncated after \( M \) terms, resulting in a system of \( 2M \) simultaneous linear algebraic equations in the truncated form of Eq. (12). This matrix equation can be numerically solved to yield the \( 2M \) unknown constants \( R_m \) and \( \hat{R}_m \) required in the truncated form of Eqs. (10) and (11) for the temperature distribution. The accuracy of the boundary collocation technique can be improved to a satisfactory degree by taking a sufficiently large value of \( M \).
2.2 Fluid Velocity Distribution

Because of the low Reynolds number encountered in thermocapillary motions, the fluid motion is governed by the following equations for axisymmetric creeping flows:

\[ E^2(\Psi) = 0 \quad (r \geq a), \]
\[ E^2(\hat{\Psi}) = 0 \quad (r \leq a), \]

where \( \Psi \) and \( \hat{\Psi} \) are the stream functions for the flow inside the drop and for the external flow, respectively, which are related to the non-vanishing components of the corresponding fluid velocities in cylindrical coordinates by

\[ (v_\rho, \hat{v}_\rho) = \frac{1}{\rho} \frac{\partial (\Psi, \hat{\Psi})}{\partial z}, \]
\[ (v_z, \hat{v}_z) = \frac{1}{\rho} \frac{\partial (\Psi, \hat{\Psi})}{\partial \rho}, \]

and \( E^2 \) is the Stokes operator.

The boundary conditions for the fluid flow at the drop surface (Young et al, 1959; Anderson, 1985), on the tube wall, and far from the drop are

\[ r = a: \quad v_\rho = \hat{v}_\rho, \]
\[ v_z = \hat{v}_z, \]
\[ v_z + v_\rho \tan \theta = U, \]
\[ \tau_{r\theta} - \hat{\tau}_{r\theta} = (-\frac{\partial \hat{\psi}}{\partial T}) \frac{\partial T}{r \partial \theta}, \]

\[ \rho = b: \quad v_\rho = v_z = 0, \]

\[ z \to \infty: \quad v_\rho = v_z = 0. \]

Here, \( \tau_{r\theta} \) and \( \hat{\tau}_{r\theta} \) are the viscous shear stresses for the external flow and the flow inside the drop, respectively, and \( U \) is the thermocapillary migration velocity of the drop in the \( z \) direction to be determined.

The general solution satisfying Eqs. (13), (14), and (18) is (Leichtberg et al., 1976)

\[ \Psi = \int_0^\infty [X(\omega) \rho I_1(\omega \rho) + Y(\omega) \rho^2 I_0(\omega \rho)] \cos(\omega z) d\omega + \sum_{n=2}^\infty \left( B_n r^{-n+1} + D_n r^{-n+3} \right) G_n^{-1/2}(\cos \theta), \]

\[ \hat{\Psi} = \sum_{n=2}^\infty (A_n r^n + C_n r^{n+2}) G_n^{-1/2}(\cos \theta), \]

where \( G_n^{-1/2} \) is the Gegenbauer polynomial of the first kind of order \( n \) and degree \(-1/2\), \( X(\omega) \) and \( Y(\omega) \) are unknown functions, \( A_n, B_n, C_n, \) and \( D_n \) are unknown constants, and \( n \) is even. Substituting Eq. (19) into Eq. (17) and applying the Fourier cosine transform on the variable \( z \) lead
to a solution for $X(\omega)$ and $Y(\omega)$ in terms of the coefficients $B_n$ and $D_n$, and then the substitution of Eqs. (19) and (20) into Eq. (15) yields the fluid velocity components as

$$v_r = \sum_{n=2}^{\infty} [B_n \gamma_{1n}(r, \theta) + D_n \gamma_{2n}(r, \theta)],$$  

$$v_\theta = \sum_{n=2}^{\infty} [B_n \gamma_{2n}(r, \theta) + D_n \gamma_{2n}(r, \theta)],$$  

$$\nabla_r = \sum_{n=2}^{\infty} [A_n \alpha_{1n}(r, \theta) + C_n \alpha_{2n}(r, \theta)],$$  

$$\nabla_\theta = \sum_{n=2}^{\infty} [A_n \alpha_{2n}(r, \theta) + C_n \alpha_{2n}(r, \theta)],$$

where the definitions of the functions $\alpha_{in}^{(i)}(r, \theta)$ and $\gamma_{in}^{(i)}(r, \theta)$ for $i$ and $j$ equal to 1 or 2 are given by Eqs. (B.5)-(B.8) in Appendix B.

The only boundary conditions that remain to be satisfied are those on the drop surface. Substituting Eqs. (21) and (22) into Eq. (16), we obtain

$$\sum_{n=2}^{\infty} [B_n \gamma_{1n}(a, \theta) + D_n \gamma_{2n}(a, \theta) - A_n \alpha_{1n}(a, \theta) - C_n \alpha_{2n}(a, \theta)] = 0,$$

$$\sum_{n=2}^{\infty} [B_n \gamma_{1n}^*(a, \theta) + D_n \gamma_{2n}^*(a, \theta) - A_n \alpha_{1n}^*(a, \theta) - C_n \alpha_{2n}^*(a, \theta)] = 0,$$

$$\sum_{n=2}^{\infty} [B_n \gamma_{1n}(a, \theta) \tan \theta + \gamma_{1n}^*(a, \theta)] + D_n [\gamma_{1n}(a, \theta) \tan \theta + \gamma_{2n}^*(a, \theta)] = U,$$

$$\sum_{n=2}^{\infty} [B_n \gamma_{1n}(a, \theta) + D_n \gamma_{2n}(a, \theta) - \eta^* A_n \alpha_{1n}^*(a, \theta) - \eta^* C_n \alpha_{2n}^*(a, \theta)]$$

$$= (\frac{\partial \gamma}{\partial T}) \frac{E_n}{\eta} \sum_{m=1}^{M} R_m \delta_{m}^{(i)}(a, \theta) - \sin \theta],$$

where $\eta^* = \hat{\eta}/\eta$, $n$ is even, $m$ is odd, and the functions $\delta_{m}^{(i)}(r, \theta)$, $\alpha_{in}^*(r, \theta)$, and $\gamma_{in}^*(r, \theta)$ for $i = 1$ or 2 are defined by Eqs. (B.3), (B.15), and (B.16). The first $M$ coefficients $R_m$ have been determined through the procedure given in the previous subsection.

Equation (23) can be satisfied by utilizing the boundary collocation technique presented for the solution of the temperature field. Along a quarter-circular longitudinal arc at the drop surface, Eq. (23) is applied at $N$ discrete points (from $\theta = 0$ to $\theta = \pi/2$) and the infinite series in Eqs. (21) and (22) are truncated after $N$ terms. This generates a set of $4N$ linear algebraic equations for the $4N$ unknown constants $A_n$, $B_n$, $C_n$, and $D_n$. The fluid velocity field is completely obtained once these coefficients are solved for a sufficiently large number of $N$. 

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2.3 Derivation of the Drop Velocity

The hydrodynamic drag force exerted on the drop can be determined from (Leichtberg et al., 1976)

\[ F = 4\pi\eta D_2. \]  

(24)

Since the drop is freely suspended in the surrounding fluid, the net force acting on the drop must vanish. Applying this constraint to Eq. (24), one has

\[ D_2 = 0. \]  

(25)

To determine the thermocapillary migration velocity \( U \) of the drop, Eq. (25) and the \( 4N \) algebraic equations resulting from Eq. (23) are to be solved simultaneously. Similar to the thermocapillary migration velocity of an unconfined drop given by Eqs. (1) and (2), the value of \( U \) is proportional to the quantity \( (-\partial\gamma/\partial T)(a/\eta) \) and dependent on the dimensionless parameters \( k^* \) and \( \eta^* \) (in addition to the radius ratio \( a/b \)).

3. Results and Discussion

The solution for the axisymmetric thermocapillary motion of a fluid sphere in a circular tube, obtained by using the boundary collocation method described in the previous section, is presented in this section. The system of linear algebraic equations to be solved for the coefficients \( R_m \) and \( \hat{R}_m \) is constructed from Eq. (12), while that for \( A_n, B_n, C_n, \) and \( D_n \) is composed of Eq. (23). All the numerical integrations to evaluate the functions \( \delta_m^{(i)}, \gamma_m^{(i)}, \) and \( \gamma_m^{(i)} \) were done by the 60-point Gauss-Laguerre quadrature.

When selecting the points along the quarter-circular generating arc of the spherical drop where the boundary conditions are satisfied, the first point that should be chosen is \( \theta = \pi/2 \), since this point defines the projected area of the drop normal to the direction of migration and controls the gap between the drop and tube wall. In addition, the point \( \theta = 0 \) is also important. However, examinations of the systems of linear algebraic equations (12) and (23) show that the resulting matrix equations become singular if these points are used. To avoid this difficulty, these points are replaced by closely adjacent points \( \theta = \pi/2 - \delta \) and \( \theta = \delta \). Additional points are selected to divide the quarter-circular arc into equal segments. A reasonable value of \( \delta \) used in this work is 0.01° (Leichtberg et al., 1976), with which the numerical results of the drop velocity converge satisfactorily.

The solutions for the thermocapillary migration velocity of a spherical drop along the axis of a tube for the cases of a wall with the prescribed far-field temperature distribution and an insulated wall are presented in Tables 1 and 2, respectively, for various values of the parameters \( k^*, \eta^*, \) and \( a/b \), in which the velocity of an identical drop in an infinite fluid given by Eqs. (1) and (2) is used to normalize the wall-corrected values. The limiting case of \( k^* = \eta^* = 0 \) denotes the system of a gas bubble in a liquid. The results of the normalized velocity \( U/U_0 \) obtained by the boundary collocation method converge satisfactorily to at least the significant figures shown in the tables, with the
accuracy and convergence behavior depending on the radius ratio $a/b$. For the most difficult case with $a/b = 0.99$, the numbers of collocation points $M = 66$ and $N = 66$ are sufficiently large to achieve this convergence.

In Appendix A, an asymptotic solution for the same confined thermocapillary migration is also obtained by using a method of reflections. The drop velocity is given by Eq. (A.20), which is a power series of $\lambda$ ($= a/b$) up to $O(\lambda^3)$. The values of the normalized drop mobility $U/U_0$ calculated from this approximate solution are also listed in Tables 1 and 2 for comparison. It can be seen that the formula (A.20) is in good agreement with the collocation results as long as $\lambda \leq 0.5$; the errors in all cases are less than 2%. However, its accuracy deteriorates rapidly, as expected, when the relative spacing between the drop and tube wall becomes small.

The collocation solutions for the normalized velocity $U/U_0$ of a fluid sphere undergoing thermocapillary motion along the axis of a circular tube as a function of the radius ratio $a/b$ are depicted in Figs. 2-4 for various values of $*k^*$ and $*\eta^*$ up to 100 for wider ranges. As expected, the drop migrates with the velocity that would exist in the absence of the wall as $a/b$ goes to 0, but the wall effect on the drop velocity can be significant as $a/b$ increases. For given values of $*\eta^*$ and $a/b$, the normalized thermocapillary mobility $U/U_0$ increases with an increase in $*k^*$ for the case of a tube wall prescribed with the far-field temperature distribution (the boundary condition (7) is used), but decreases with an increase in $*k^*$ for the case of an insulated wall (the boundary condition (8) is used), because the temperature gradients on the drop surface near a wall with the imposed far-field temperature gradient increase as the relative conductivity $*k^*$ increases and these gradients near an insulated wall decrease as $*k^*$ increases.

For any set of fixed values of $*k^*$ and $*\eta^*$, different from the corresponding migration between two plane walls (Keh et al., 2002), the normalized thermocapillary mobility $U/U_0$ of a fluid sphere along the axis of a tube decreases monotonically with an increase in $a/b$ and almost vanishes in the limit of calculations ($a/b = 0.99$), although the reflection result in Eq. (A.20) indicates the possibility of wall enhancement on the thermocapillary motion as $*\eta^*$ is large and $a/b$ is close to unity (for the case of an insulated wall with a small value of $*k^*$ or a wall prescribed with the far-field temperature distribution with a large value of $*k^*$). For the particular case of $*k^* = 1$, the two types of tube wall will result in the same thermocapillary motion (the effect of thermal interaction between the drop and wall disappears) and the relative thermocapillary mobility of the drop decreases with $a/b$ solely because of the hydrodynamic resistance exerted by the tube wall.
Table 1 Normalized thermocapillary migration velocity of a spherical drop along the axis of a circular tube whose wall is prescribed with the far-field temperature profile from the exact boundary-collocation solution and asymptotic method-of-reflection solution.

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>$U/U_0$</th>
<th>$\eta^*=0$</th>
<th>$\eta^*=1$</th>
<th>$\eta^*=10$</th>
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</thead>
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<td></td>
<td>Exact</td>
<td>Asymptotic</td>
<td>Exact</td>
<td>Asymptotic</td>
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<tr>
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Table 2 Normalized thermocapillary migration velocity of a spherical drop along the axis of a circular tube with an insulated wall from the exact boundary-collocation solution and asymptotic method-of-reflection solution.

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On the other hand, the normalized thermocapillary mobility $U/U_0$ of the drop increases with an increase in $\eta^*$ for any specified values of $k^*$ and $a/b$ in both cases of the tube wall, in agreement with the predictions from the method-of-reflection solution given by Eq. (A.20) and by Mahesri et al. (2014), but opposite to the outcome that $U/U_0$ decreases with an increase in $\eta^*$ obtained by Chen et al. (1991) for an insulated tube wall. Therefore, the numerical solutions of Chen et al. that $U/U_0$ decreases with an increase in $\eta^*$ under the same condition as the current work should be in error.

![Graph of normalized thermocapillary mobility](image)

**Fig. 2.** Plots of the normalized thermocapillary mobility $U/U_0$ of a spherical drop along the axis of a circular tube versus the radius ratio $a/b$ for various values of $\eta^*$: (upper) $k^* = 0$; (lower) $k^* = 100$. The solid curves represent the case of a tube wall with the far-field temperature distribution and the dashed curves denote the case of an insulated tube wall.
Fig. 3. Plots of the normalized thermocapillary mobility $U/U_0$ of a drop with $\eta^* = 1$ versus $a/b$ for various values of $k^*$. The solid and dashed curves denote the case of a tube wall with the far-field temperature distribution and the case of an insulated tube wall, respectively.

Fig. 4. Plots of the normalized thermocapillary mobility (solid curves, with $k^* = 1$) and settling mobility (dashed curves) of a spherical drop in a tube versus $a/b$ for various values of $\eta^*$.

For the creeping motion of a fluid sphere on which a constant body force (such as gravitational force) $F_0$ is exerted along the axis of a circular tube, the numerical result of the drop velocity was obtained by using the boundary collocation method (Keh and Chang, 2007). A comparison of the wall effects on the migration of the drop under gravity (where $U_0 = (F/6\pi\eta a)(3\eta^* + 3)/(3\eta^* + 2)$) and on the
thermocapillary motion is given in Fig. 4. Evidently, the boundary effect on the thermocapillary migration is weaker than that on a settling or buoyantly rising drop (which is discussed after Eq. (A.13) in Appendix A). Note that the boundary effect on the drop migration in a gravitational field is stronger when the value of $\eta^*$ becomes greater, opposite to that which would occur if the drop migrates because of thermocapillarity.

4. Conclusions

In this paper, the exact numerical solutions and asymptotic analytical solutions for the thermocapillary migration of a spherical drop along the axis of a circular tube are obtained by using the methods of boundary collocation and reflections, respectively. Both the cases of a tube wall prescribed with the far-field temperature distribution and an insulated tube wall are studied. It is found that the wall effect on the thermocapillary migration is significant and the agreement between the collocation solution and the reflection solution is good. For fixed values of the relative thermal conductivity $k^*$ and viscosity $\eta^*$, the normalized thermocapillary mobility $U/U_0$ of a drop in a tube is a decreasing function of the radius ratio $a/b$. This outcome reflects the dominance of the hydrodynamic resistance acting on the drop movement by the tube wall over the possible mobility enhancement due to the drop-wall thermal interaction. $U/U_0$ increases/decreases with an increase in $k^*$ for the case of a conducting/insulating tube wall and increases with an increase in $\eta^*$.

The thermocapillary mobility of a fluid sphere along the middle plane between two parallel plane walls has been calculated for various values of $k^*$, $\eta^*$, and $a/b$, where $b$ for this geometry is the distance between the drop center and either plane wall (Keh et al., 2002). For the case that the two plane walls are imposed with the far-field temperature distribution and $k^*$ is large, the drop mobility was found to first decrease and then increase with increases in $a/b$. When the gap between the drop and a plane wall turns thin, the drop can even move faster than it would if $a/b = 0$. Similar dependence of $U/U_0$ on $a/b$ was also obtained for the case of insulated plane walls with small $k^*$.

The difference between the boundary effects on thermocapillary motion in a tube ($U/U_0$ is a monotonic decreasing function of $a/b$) and in a slit is surprising, reflecting the much stronger effect of viscous interactions in a tube than in a slit. Because a slit confines the drop only from its top and bottom, not from all directions as a tube does, the net boundary effect on thermocapillary motion of a drop is generally stronger in a tube than in a slit.

Appendix A. Analysis of the Thermocapillary Migration of a Fluid Sphere in a Circular Tube by a Method of Reflections

The thermocapillary motion of a spherical drop with the relative viscosity $\eta^*$ and thermal conductivity $k^*$ along the axis of a circular tube under the applied temperature gradient $E_e e_z$, as illustrated in Fig. 1, will be analyzed in this appendix by using a method of reflections. The solutions for the temperature and velocity distributions in the external fluid can be expressed as expansion
series relating to increasing powers of the drop-to-tube radius ratio $\lambda = a / b$:

$$T = T_0 + E_a z + T_p^{(1)} + T_w^{(1)} + T_p^{(2)} + T_w^{(2)} + \ldots, \quad \text{(A.1)}$$

$$\mathbf{v} = \mathbf{v}_p^{(1)} + \mathbf{v}_w^{(1)} + \mathbf{v}_p^{(2)} + \mathbf{v}_w^{(2)} + \ldots, \quad \text{(A.2)}$$

where the subscripts $p$ and $w$ denote the reflections from the fluid drop and tube wall, respectively, the superscript $(i)$ represents the $i$th reflections from the drop and wall surfaces, and all the temperature and velocity expansion terms are governed by Eqs. (3) and (13), respectively. Thus, the thermocapillary migration velocity of the drop has the form

$$U = U_0 e_z + U_p^{(1)} + U_w^{(2)} + \ldots, \quad \text{(A.3)}$$

where $U_0 = M_T E_\alpha$ is the velocity of an unconfined drop given by Eqs. (1) and (2), and the expansion terms $U^{(i)}$ are related to $T^{(i)}$ and $v^{(i)}$ by the Faxen law (Anderson, 1985; Chen and Keh, 1990; Choudhuri and Raja Sekhar, 2013)

$$U^{(i)} = [M_T \nabla T^{(i)} + v^{(i)} + C \frac{a}{6} \nabla^2 v^{(i)}]_{r=0}, \quad \text{(A.4)}$$

with $C = 3\eta^* / (2 + 3\eta^*) \ (0 \leq C \leq 1)$.

The first reflected temperature and velocity distributions from the drop satisfying the boundary conditions (5) and (16) are

$$T_p^{(1)} = G E_a a^3 r^{-2} \cos \theta, \quad \text{(A.5)}$$

$$v_p^{(1)} = \frac{1}{2} U_0 a^3 r^{-3} (2 \cos \theta r_e + \sin \theta r_e), \quad \text{(A.6)}$$

where $G = (1 - k^*)(2 + k^*)^{-1} \ (-1 \leq G \leq 1/2 )$. For a tube with insulated wall, the boundary conditions for the $i$th reflected fields from the wall consistent with Eqs. (6), (8), (17), and (18) are

$$\rho = b : \quad \frac{\partial T^{(i)}}{\partial \rho} = - \frac{\partial T^{(i)}_p}{\partial \rho}, \quad \text{(A.7a)}$$

$$v^{(i)} = -v_p^{(i)}; \quad \text{(A.7b)}$$

$$|z| \to \infty : \quad T^{(i)}_w \to 0, \quad v^{(i)}_w \to 0. \quad \text{(A.8a)}$$

The solution of $T^{(1)}_w$ can be obtained by applying the Fourier sine transform on the variable $z$ in Eqs. (3), (A.7a), and (A.8a) (taking $i = 1$), which results in

$$T^{(1)}_w = \frac{2}{\pi} G E_a \lambda^2 \int_0^\infty \frac{K_1(\omega)}{I_1(\omega)} I_0 \left(\frac{\rho}{b}\omega\right) \sin \left(\frac{\lambda}{b} \omega \right) d\omega, \quad \text{(A.9)}$$

where $I_n(\omega)$ and $K_n(\omega)$ are the modified Bessel functions of the first and second kinds, respectively, of order $n$. The solution of $v^{(1)}_w$ is obtained by applying the Fourier cosine transform twice to Eqs. (13), (A.7b), and (A.8b),
 Han C. Chiu and Huan J. Keh / American Journal of Heat and Mass Transfer  
(2016) Vol. 3 No. 1 pp. 15-36

\[ \mathbf{v}_w^{(1)} = \frac{U_0 \lambda^3}{\pi} \int_0^\infty \left[ \beta_1(\omega) - \beta_2(\omega) \right] \sin(\frac{z}{\lambda} \omega) \mathbf{e}_\rho \]  
\[ \text{where} \]
\[ \beta_1(\omega) = \left[ I_1(\omega) \right]^2 - I_0(\omega) I_2(\omega) \]
\[ \beta_2(\omega) = \omega^2 [ I_1(\omega) K_1(\omega) + I_0(\omega) K_2(\omega) ] \beta_1(\omega). \]

Using Eq. (A.4), we determine the contributions of \( T_w^{(1)} \) and \( \mathbf{v}_w^{(1)} \) to the drop velocity as

\[ \mathbf{U}_1^{(1)} = M_1 \nabla T_w^{(1)} \bigg|_{r=0} = d_1 G \lambda^3 U_0 \mathbf{e}_z, \]  
\[ \mathbf{U}_h^{(1)} = [\mathbf{v}_w^{(1)} + C \frac{\partial^2}{\partial \mathbf{z}^2} \mathbf{v}_w^{(1)}] \bigg|_{r=0} = (d_2 \lambda^3 + d_3 C \lambda^3) U_0 \mathbf{e}_z ; \]
\[ \mathbf{U}^{(1)} = \mathbf{U}_1^{(1)} + \mathbf{U}_h^{(1)} = \lambda^3 (d_1 G + d_2 + d_3 C \lambda^3) U_0 \mathbf{e}_z, \]

where

\[ d_1 = \frac{2}{\pi} \int_0^\infty \frac{K_1(\omega)}{I_1(\omega)} \omega^2 \mathbf{d} \omega = 1.59365, \]  
\[ d_2 = \frac{1}{\pi} \int_0^\infty \left[ 2 \beta_1(\omega) - \beta_2(\omega) \right] \mathbf{d} \omega = -2.08669, \]  
\[ d_3 = \frac{1}{3\pi} \int_0^\infty \beta_1(\omega) \omega^2 \mathbf{d} \omega = 1.89632. \]

Equation (A.12a) indicates that the reflected temperature field from the insulated tube wall can increase (if \( k^* < 1 \) or \( G > 0 \)) or decrease (if \( k^* > 1 \) or \( G < 0 \)) the migration velocity of the drop from its undisturbed value, while Eq. (A.12b) shows that the reflected velocity field is to decrease this velocity; the net effect of the reflected fields is expressed by Eq. (A.13), which can enhance or retard the movement of the drop, depending on the values of \( \eta^* \), \( k^* \) (or \( G \)), and \( \lambda \). When \( k^* = 1 \) (or \( G = 0 \)), the reflected temperature field makes no contribution to the drop velocity. For the case of insulated tube wall with any given values of \( \eta^* \) and \( \lambda \), the normalized drop velocity decreases monotonically with an increase in \( k^* \). Equation (A.13) illustrates that the wall correction to the thermocapillary migration velocity of the drop is \( \mathcal{O}(\lambda^3) \), which is weaker than that obtained for the corresponding sedimentation problem, in which the leading boundary effect is \( \mathcal{O}(\lambda) \). The wall effect on the thermocapillary migration involving \( \eta^* \) starts from \( \mathcal{O}(\lambda^5) \), and the normalized drop velocity increases with an increase in \( \eta^* \).

The solution for the second reflected fields from the drop is

\[ T_p^{(2)} = E_w [d_1 G \lambda^3 a^5 r^{-2} \cos \theta + \mathcal{O}(\lambda^5 a^5)], \]  
\[ \mathbf{v}_p^{(2)} = \frac{1}{2} U_0 [d_1 G \lambda^3 a^5 r^{-3} (2 \cos \theta \mathbf{e}_r + \sin \theta \mathbf{e}_\theta) + \mathcal{O}(\lambda^5 a^3)]. \]

The boundary conditions for the second reflected fields from the wall are obtained by substituting the
results of $T_p^{(2)}$ and $v_p^{(2)}$ into Eqs. (A.7) and (A.8), with which Eqs. (3) and (13) can be solved as before to yield
\[
[\nabla T_w^{(2)}]_{r=0} = [d_1^2 G^2 \lambda^6 + O(\lambda^9)]E_{z} e_z, \quad \text{(A.17)}
\]
\[
[v_w^{(2)} + C \frac{G^2}{6} \nabla^2 v_w^{(2)}]_{r=0} = [d_1 d_2 G \lambda^6 + d_1 d_2 CG \lambda^6 + O(\lambda^9)]U_{0} e_z. \quad \text{(A.18)}
\]
The contribution of the second reflected fields to the drop velocity is obtained by substituting Eqs. (A.17) and (A.18) into Eq. (A.4),
\[
U^{(2)} = [d_1 G \lambda^6 (d_1 G + d_2 + d_3 C \lambda^2) + O(\lambda^9)]U_{1} e_z. \quad \text{(A.19)}
\]
With the substitution of Eqs. (A.13) and (A.19) into Eq. (A.3) and knowing that $U^{(3)} = O(\lambda^9)$, the drop velocity can be expressed as $U = U_{0} e_{z}$ with
\[
U = U_{0} [1 + \lambda^* (1 + d_1 G \lambda^3) (d_1 G + d_2 + d_3 C \lambda^2) + O(\lambda^9)]. \quad \text{(A.20)}
\]
The necessary condition in the first two reflections for the wall enhancement on the thermocapillary migration to occur is a small value of $k^*$, a large value of $\eta^*$, and a value of $\lambda$ close to unity such that the relation $d_3 C \lambda^2 > -(d_1 G + d_2)$ is warranted.

For the case that a linear temperature profile consistent with the far-field distribution is prescribed on the tube wall, namely, the boundary condition (8) is replaced by Eq. (7), the series expansions (A.1)-(A.3), the solutions of $T_p^{(i)}$ and $v_p^{(i)}$ in Eqs. (A.5) and (A.6), and the boundary conditions for $T_w^{(i)}$ and $v_w^{(i)}$ in Eqs. (A.7b) and (A.8) are still valid, while Eq. (A.7a) becomes
\[
\rho = b: \quad T_w^{(i)} = - T_p^{(i)}. \quad \text{(A.21)}
\]
The solution of $T_w^{(i)}$ satisfying Eqs. (3), (A.21), and (A.8a) is
\[
T_w^{(i)} = - \frac{2}{\pi} G E_{z} a \lambda^2 \int_{0}^{\infty} \omega \frac{K_0(\omega)}{I_0(\omega)} \frac{P}{b^3} \omega \sin \left( \frac{\omega}{b} \right) \omega \ d\omega, \quad \text{(A.22)}
\]
whereas the solution of $v_w^{(i)}$ is unchanged from Eqs. (A.10) and (A.11). The results of the following reflected fields and of the drop velocity are also obtained from Eqs. (A.12)-(A.20) by replacing $d_1$ by $\overline{d_1}$, where
\[
\overline{d_1} = - \frac{2}{\pi} \int_{0}^{\infty} \omega^2 \frac{K_0(\omega)}{I_0(\omega)} d\omega = -0.411822. \quad \text{(A.23)}
\]
Contrary to the effect of an insulated tube wall, the reflected temperature field from a tube wall with the prescribed far-field temperature distribution reduces the velocity of the drop if $k^* < 1$ or $G > 0$ and enhances this velocity if $k^* > 1$ or $G < 0$ (the normalized drop velocity increases monotonically with an increase in $k^*$ for given values of $\eta^*$ and $\lambda$). When $k^* = 1$ or $G = 0$, the two types of tube wall will produce the same effects on the thermocapillary motion of the drop (the thermal interaction between the drop and wall disappears). Under the condition that the values of $k^*$, $\eta^*$,
and $\lambda$ are sufficiently large such that $d_3 C \lambda^2 > -(\vec{d}_1 G + d_2)$, the net effect of a tube wall prescribed with the far-field temperature distribution can enhance the thermocapillary migration of a drop in the first two reflections.

**Appendix B. Definitions of Some Functions in Section 2**

The functions $\delta_m^{(1)}, \delta_m^{(2)},$ and $\delta_m^{(3)}$ in Eqs. (11), (12), and (23d) are defined by

$$\delta_m^{(1)}(r, \theta) = \frac{\partial}{\partial r} \delta_m^{(1)} / \partial \theta = \int_0^\infty \omega H_m(\omega) [I_0(\omega r \sin \theta) \cos(\omega r \cos \theta) \cos \theta + I_1(\omega r \sin \theta) \sin(\omega r \cos \theta) \sin \theta] \, d\omega - (m+1)r^{-m-2} P_m(\cos \theta),$$

(B.1)

where

$$H_m(\omega) = (-1)^{m-2}\frac{2\omega^m \sin^m \theta}{\pi m! \sin \theta} K_v(b \omega),$$

(B.4)

$v = 0$ if Eq. (7) is used for the boundary condition of the temperature field at the tube wall and $v = 1$ if Eq. (B.8) is used.

The functions $\alpha_m^{(i)}$ and $\gamma_m^{(i)}$ for $i$ and $j$ equal to 1 or 2 in Eqs. (21)-(23c) are

$$\alpha_m^{(1)}(r, \theta) = -r^{n+2i-4} [(n+1)G_{n+1/2}(\cos \theta) \csc \theta - (2n+2i-3)G_{n-1/2}(\cos \theta) \cot \theta],$$

(B.5)

$$\alpha_m^{(2)}(r, \theta) = -r^{n+2i-4} [(2n+2i-3)G_{n+1/2}(\cos \theta) + P_n(\cos \theta)],$$

(B.6)

$$\gamma_m^{(1)}(r, \theta) = \int_0^\infty \left[ S_{ni}^{(1)}(\omega) I_0(\omega r \sin \theta) \omega r \sin \theta + S_{ni}^{(2)}(\omega) I_1(\omega r \sin \theta) \right] \sin(\omega r \cos \theta) \, d\omega$$

$$- r^{n+2i-3} [(n+1)G_{n+1/2}(\cos \theta) \csc \theta - 2(i-1)G_{n-1/2}(\cos \theta) \cot \theta],$$

(B.7)

$$\gamma_m^{(2)}(r, \theta) = \int_0^\infty \left[ S_{ni}^{(1)}(\omega) [2I_0(\omega r \sin \theta) + I_1(\omega r \sin \theta) \omega r \sin \theta] + S_{ni}^{(2)}(\omega) I_0(\omega r \sin \theta) \right]$$

$$\times \cos(\omega r \cos \theta) \, d\omega - r^{n+2i-3} [P_n(\cos \theta) + 2(i-1)G_{n+1/2}(\cos \theta)].$$

(B.8)

In Eqs. (B.7) and (B.8),

$$S_{ni}^{(1)}(\omega) = A_{ni}^{(1)}(\omega)I_0(b \omega) - A_{ni}^{(2)}(\omega) I_1(b \omega)$$

$$\left[ b \omega I_0(b \omega) I_2(b \omega) - I_1(b \omega) \right]^2,$$

(B.9)

$$S_{ni}^{(2)}(\omega) = \frac{A_{ni}^{(1)}(\omega) - b \omega I_0(b \omega) S_{ni}^{(1)}(\omega)}{I_1(b \omega)},$$

(B.10)

where
\begin{align}
A_{1n}^{(1)}(\omega) &= (-1)^{n/2} \frac{2\omega^n}{\pi n!} K_n(b\omega), \\
A_{2n}^{(1)}(\omega) &= (-1)^{n/2} \frac{2\omega^{n-2}}{\pi n!} [(n-2)(n-3)K_1(b\omega) - (2n-3)\omega K_0(b\omega)], \\
A_{1n}^{(2)}(\omega) &= (-1)^{n/2} \frac{2\omega^n}{\pi n!} K_0(b\omega), \\
A_{2n}^{(2)}(\omega) &= (-1)^{n/2} \frac{2\omega^{n-2}}{\pi n!} [(2n-3)\omega K_1(b\omega) - n(n-1)K_0(b\omega)].
\end{align}

The functions $\alpha_n^*$ and $\gamma_n^*$ for $i$ equal to 1 or 2 in Eq. (23d) are defined by

\begin{align}
\alpha_n^*(r, \theta) &= -r^{n+2-5} [(n+1)(n+2i-5)G_{n+1}^{-1/2}(\cos \theta) \cot \theta \\
&- (n+2i-5)(2n+2i-3)G_n^{-1/2}(\cos \theta) \csc \theta \\
&+ (5-2i+n \cot^2 \theta) P_n(\cos \theta) \sin \theta - nP_{n-1}(\cos \theta) \cot \theta], \\
\gamma_n^*(r, \theta) &= -\cos \theta \sin \theta [C_n^*(r, \theta) + D_n^*(r, \theta)] - (\cos^2 \theta - \sin^2 \theta)[C_n^{**}(r, \theta) + D_n^{**}(r, \theta)],
\end{align}

where

\begin{align}
C_1(r, \theta) &= -2r^{-n+1} [(n+1)(n+\csc^2 \theta)G_n^{-1/2}(\cos \theta) - (3n+2)P_n(\cos \theta) \cos \theta \\
&+ nP_{n-1}(\cos \theta)]
\end{align}

\begin{align}
C_{2n}^*(r, \theta) &= 2r^{-n} [2(2n-1+\cot^2 \theta)G_n^{-1/2}(\cos \theta) \cos \theta - (n+1)(n-1+\cot^2 \theta)G_n^{-1/2}(\cos \theta) \\
&- (n+2-4\sin^2 \theta)P_{n-1}(\cos \theta) + 3nP_n(\cos \theta) \cos \theta], \\
C_1^{**}(r, \theta) &= -r^{-n+2} [n \cot \theta(n+1)G_n^{-1/2}(\cos \theta) + P_{n-1}(\cos \theta)] \\
&+ [(3n+2) \sin \theta - n \csc \theta]P_n(\cos \theta)
\end{align}

\begin{align}
C_{2n}^{**}(r, \theta) &= -r^{-n} [2(2n-1)\sin \theta - (n-2)\csc \theta]G_n^{-1/2}(\cos \theta) \\
&+ (n^2 - n - 2)G_n^{-1/2}(\cos \theta) \cot \theta + (n-4\sin^2 \theta)P_{n-1}(\cos \theta) \cot \theta \\
&+ n(3\sin \theta - \csc \theta)P_n(\cos \theta)
\end{align}

\begin{align}
D_n^*(r, \theta) &= -2\int_0^\infty \{S_i^{(1)}(\omega)[3I_0(\omega \sin \theta) + 2I_1(\omega \sin \theta) \omega \sin \theta] \\
&+ S_i^{(2)}(\omega)[2I_0(\omega \sin \theta) - I_1(\omega \sin \theta) \csc \theta / \omega]\} \omega \sin(\omega \cos \theta) \text{d}\omega,
\end{align}

\begin{align}
D_n^{**}(r, \theta) &= -2\int_0^\infty \{S_i^{(1)}(\omega)[I_1(\omega \sin \theta) + I_0(\omega \sin \theta) \omega \sin \theta] \\
&+ S_i^{(2)}(\omega)I_1(\omega \sin \theta) \omega \cos(\omega \cos \theta) \text{d}\omega.
\end{align}
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Conflict of Interest

None

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