Exact Analytic Heat Transfer from an Annular Fin with Stepped Rectangular Profile

Antonio Campo\(^1\)* and Balaram Kundu\(^2\)

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Abstract

In the present study, an exact analytical methodology is developed to determine the temperature distributions, tip temperatures, heat transfer rates and fin efficiencies for annular fins with stepped rectangular profiles assuming 1) constant thermal conductivity, 2) uniform convective heat transfer coefficient in the neighboring fluid and 3) variable step radius. The two quasi one-dimensional heat conduction equations for the thick and thin parts in annular fins with stepped rectangular profiles subject to the appropriate boundary conditions are solved with modified Bessel functions of first and second kind of order zero and one. The thermo-geometric parameter chosen to oversee the heat transfer analysis is the Biot number based on the inner radius, which is also related to the Biot number based on the thick part.

Keywords: Annular fin with stepped rectangular profile; Exact analytical method; Modified Bessel functions; Two-part temperature distribution; Heat transfer rate; Fin efficiency

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>dimensionless parameter in eq. (6a)</td>
</tr>
<tr>
<td>( A_b )</td>
<td>base area (m(^2))</td>
</tr>
<tr>
<td>( Bi_i )</td>
<td>Biot number based on the inner radius, ( h r_i / k )</td>
</tr>
<tr>
<td>( Bi_{i\delta} )</td>
<td>Biot number based on the thick semi-thickness, ( h \delta_i / k )</td>
</tr>
<tr>
<td>( h )</td>
<td>mean convective heat transfer coefficient (W m(^{-2}) K(^{-1}))</td>
</tr>
<tr>
<td>( I_m(x) )</td>
<td>modified Bessel function of first kind of order ( m ) and argument ( x )</td>
</tr>
<tr>
<td>( k )</td>
<td>thermal conductivity of the material (Wm(^{-1}) K(^{-1}))</td>
</tr>
</tbody>
</table>

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**1. Introduction**

Annular fins of rectangular profile are frequently attached to circular pipes to increase the surface area, and consequently, to enhance the heat transfer rate between the pipe surface and the surrounding fluid. Annular fins are widely used in many engineering applications which include air conditioning, refrigeration, automobile, aerospace, chemical processing equipment, etc. Annular fins with rectangular profile are a common choice and their fabrication is relatively standard. The work done on annular fins with various profiles until the year 2000 has been comprehensively reviewed in a treatise by Kraus et al. (2002).

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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_m(x)$</td>
<td>modified Bessel function of second kind of order $m$ and argument $x$</td>
</tr>
<tr>
<td>$q$</td>
<td>actual heat transfer rate (W)</td>
</tr>
<tr>
<td>$q_{\text{ideal}}$</td>
<td>ideal heat transfer rate (W)</td>
</tr>
<tr>
<td>$Q$</td>
<td>dimensionless actual heat transfer rate, $q/[4\pi k r(T_b - T_f)]$</td>
</tr>
<tr>
<td>$Q_{\text{ideal}}$</td>
<td>dimensionless ideal heat transfer rate, $q_{\text{ideal}}/[4\pi k r(T_b - T_f)]$</td>
</tr>
<tr>
<td>$r$</td>
<td>radial coordinate (m)</td>
</tr>
<tr>
<td>$r_1$</td>
<td>inner radius (m)</td>
</tr>
<tr>
<td>$r_2$</td>
<td>step radius (m)</td>
</tr>
<tr>
<td>$r_3$</td>
<td>outer radius (m)</td>
</tr>
<tr>
<td>$R$</td>
<td>dimensionless radial coordinate, $r/r_3$</td>
</tr>
<tr>
<td>$R_1$</td>
<td>dimensionless inner radius, $r_1/r_3$</td>
</tr>
<tr>
<td>$R_2$</td>
<td>dimensionless step radius, $r_2/r_3$</td>
</tr>
<tr>
<td>$T_b$</td>
<td>base temperature (K)</td>
</tr>
<tr>
<td>$T_f$</td>
<td>fluid temperature (K)</td>
</tr>
<tr>
<td>$T_1$</td>
<td>temperature for thick part (K)</td>
</tr>
<tr>
<td>$T_2$</td>
<td>temperature for thin part (K)</td>
</tr>
<tr>
<td>$Z_0, Z_1, Z_2$</td>
<td>dimensionless parameters in eq. (3)</td>
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Greek letters:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1, \ldots, \gamma_5$</td>
<td>dimensionless parameters in eq. (6)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>thickness ratio, $\delta_2/\delta_1$</td>
</tr>
<tr>
<td>$\delta_1$</td>
<td>thick semi-thickness (m)</td>
</tr>
<tr>
<td>$\delta_2$</td>
<td>thin semi-thickness (m)</td>
</tr>
<tr>
<td>$\eta$</td>
<td>fin efficiency</td>
</tr>
<tr>
<td>$\xi$</td>
<td>aspect ratio for thick part, $\delta_1/r_1$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>dimensionless temperature for thick part, $(T_1 - T_f)/(T_b - T_f)$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>dimensionless temperature for thin part, $(T_2 - T_f)/(T_b - T_f)$</td>
</tr>
</tbody>
</table>
From a historical perspective, the most important exact analytic solutions of heat conduction in annular fins of rectangular profile have been developed by Harper and Brown (1922), Murray (1938), Carrier and Anderson (1944) and Gardner (1945). These exact analytic solutions are based upon the standard assumptions of quasi one-dimensional heat transfer model, constant thermal conductivity, uniform convective heat transfer coefficient and negligible heat loss at the tip.

To enhance the heat transfer rate in annular fins farther, while keeping the rectangular profile intact, there are three possibilities that can be pursued. One possibility is to relate the optimum thickness and the optimum length to the mean convection heat transfer coefficient and the thermal conductivity of the material. Representative references of this effort are those by Jakob (1949), Brown (1965), Ullman and Kalman (1989), Heggs and Ooi (2004) and Arslanturk (2005). A second possibility is to cover the annular fin of rectangular profile with a thin layer of material with a very high thermal conductivity, so that the effective thermal conductivity of the substrate/coating is aggrandized. The material-based avenue was explored theoretically by Campo (2001) and by Waszkiewicz et al. (2009) experimentally. Owing that in general fins rejects more heat near the base and less heat near the tip, the third possibility is to reduce the thickness of the annular fin with rectangular profile removing material far from the base, between the mid part and the tip. This option is achievable utilizing an annular fin with a single stepped rectangular profile and constitutes the topic of the present study.

2. Mathematical Analysis

Consider an annular fin with stepped rectangular profile as shown in Figure 1. The inner radius for the

![Fig. 1. Cross-sectional view of an annular fin with stepped rectangular profile.](image)

thick part is \( r_1 \), the outer radius for the thin part is \( r_3 \), while the step radius is set at \( r_2 \). The thickness for the thick part is \( 2\delta_1 \) and the thickness for the thin part is \( 2\delta_2 \). The thermal conductivity of the
material $k$ is considered invariant with temperature. The mean convective heat transfer coefficient over the two exposed surfaces of the annular fin is taken as $h$.

Let the dimensionless and normalized variables for temperature and radius are:
\[ \theta = (T_i - T_f) / (T_b - T_f), \quad \phi = (T_b - T_f) / (T_b - T_f), \quad R = r / r_0 \]
so that the derived dimensionless geometric parameters are
\[ R_1 = r_1 / r_0, \quad R_2 = r_2 / r_0, \quad \xi = \delta_1 / r_1, \quad \delta = \delta_2 / \delta_1 \]
In addition, the dimensionless thermo-geometric parameter chosen is the Biot number based on the inner radius of the thick part:
\[ \text{Bi}_i = h r_1 / k \]

Typically, fins are much longer than they are thick. Because of this feature, it is common and fairly accurate to assume that the temperature varies in the lengthwise direction of fins only and is essentially uniform in the cross section (Kraus et al., 2002). However, justification of the quasi one-dimensional approximation for the analysis of annular fins of rectangular has been based only on the geometric criterion of large fin length with respect to small thickness. This “rule-of-thumb” approach does not take into consideration physical components, such as the ratio of interior conductive resistance $R_k$ to exterior convective resistances $R_c$, i.e., the transverse Biot number. Lau and Tan (1973) investigated numerically the validity of the quasi one-dimensionality in a collection of annular fins of uniform rectangular profile. The authors recommended that the transverse Biot number based on the semi-thickness $t$ must be $\text{Bi}_i = h t / k \leq 0.1$. Transferring this information to the annular fins with stepped rectangular profile under study, the quasi one-dimensionality criterion translates into $\text{Bi}_i / t = h r_1 / k \leq 0.1$ for the thick part, because the thin part with $\delta_2 < \delta_1$ is automatically satisfied.

The two governing quasi one-dimensional heat conduction equations in dimensionless form are written as follows:

1) for the thick part
\[ \frac{d^2 \theta}{d R^2} + \frac{1}{R} \frac{d \theta}{d R} - Z_1 \theta = 0 \quad \text{in the domain } R_1 \leq R \leq R_2 \]

2) for the thin part
\[ \frac{d^2 \phi}{d R^2} + \frac{1}{R} \frac{d \phi}{d R} - Z_2 \phi = 0 \quad \text{in the domain } R_2 \leq R \leq 1 \]
with the presence of two additional dimensionless parameters $Z_1$ and $Z_2$ defined by
\[ Z_0 = \sqrt{\text{Bi}_i / \xi}, \quad Z_1 = Z_0 / R_1, \quad Z_2 = Z_1 / \sqrt{\delta}, \]
in which the symbol $\xi$ in $Z_0$ represents the aspect ratio for the thick part, i.e., $\delta_1 / r_1$.

Assuming negligible heat transfer through the tip, the applicable set of boundary conditions is:

at $R = R_1$, $\theta = 1$ \hspace{1cm} (4a)

at $R = R_2$, $\theta = \phi$ \hspace{1cm} (4b)

at $R = R_0$, $d \theta / d R = d \phi / d R$ \hspace{1cm} (4c)
The two second order, ordinary differential equations (2a) and (2b) with variable coefficients are classified as modified Bessel equations (Watson, 1995). The exact analytical solutions of eqs. (2a) and (2b) subject to the boundary conditions in eqs. (4a) – (4d) are described in the forthcoming section.

3. Exact Analytical Method

The dimensionless temperature distributions in the annular fin with stepped rectangular profile are written as follows:

1) for the thick part:

\[
\partial(R) = \frac{I_0(Z_R)K_0(Z_R) - I_1(Z_R)K_0(Z_R)}{I_0(Z_R)K_0(Z_R) - I_1(Z_R)K_0(Z_R)} + \frac{\lambda I_1(Z_R)K_0(Z_R) - I_0(Z_R)K_0(Z_R)}{I_0(Z_R)K_0(Z_R) - I_1(Z_R)K_0(Z_R)} \quad \text{in the domain } R_1 \leq R \leq R_2
\]

(5a)

2) for the thin part:

\[
\phi(R) = \frac{\lambda I_1(Z_R)K_0(Z_R) - I_0(Z_R)K_0(Z_R)}{I_0(Z_R)K_0(Z_R) + I_1(Z_R)K_0(Z_R)} \quad \text{in the domain } R_2 \leq R \leq 1
\]

(5b)

In eqs. (5a) and (5b), \( I_m(x) \) denotes the modified Bessel function of first kind of order \( m = 0, 1 \) and argument \( x \) and \( K_m(x) \) denotes the modified Bessel function of second kind of order \( m = 0, 1 \) and argument \( x \). Further, an additional dimensionless parameter \( A \) showing up in eqs. (5a) and (5b) is given by the expression

\[
A = \frac{\gamma_1\gamma_5}{\gamma_2\gamma_3\gamma_4} + \delta \gamma_3\gamma_5
\]

(6a)

in which \( \delta \) is the thickness ratio already defined in eq. (1b) and the five \( \gamma \) components are given by

\[
\gamma_1 = I_1(Z_R)K_0(Z_R) + I_0(Z_R)K_1(Z_R)
\]

(6b)

\[
\gamma_2 = I_0(Z_R)K_0(Z_R) + I_1(Z_R)K_0(Z_R)
\]

(6c)

\[
\gamma_3 = I_0(Z_R)K_0(Z_R) - I_0(Z_R)K_0(Z_R)
\]

(6d)

\[
\gamma_4 = I_0(Z_R)K_1(Z_R) - I_1(Z_R)K_0(Z_R)
\]

(6e)

\[
\gamma_5 = I_0(Z_R)K_0(Z_R) + I_1(Z_R)K_0(Z_R)
\]

(6f)

4. Thermal Parameters of Interest

4.1. Tip temperature

The safe – touch temperature of hot solid bodies is an important issue for the safety of technical personnel in plant environments as cited by Arthur and Anderson (2004). Owing that fins attached to tubes in heat exchange devices are prone to be touched accidentally, the tip temperature of fins is
considered by design engineers as a “parameter of relevance”. Correspondingly, the dimensionless tip temperature in the annular fin with stepped rectangular profile \(T(r_2)\) is obtainable from eq. (5b),

\[
\phi_{tip} = \phi(1) = \frac{A[I_0(Z_2)K_0(Z_2)-I_1(Z_2)K_1(Z_2)]}{I_0(Z_2)K_1(Z_2)+I_1(Z_2)K_0(Z_2)}
\]

(7)

where \(A\) is taken from eq. (6a).

4.2. Heat transfer rate

The actual heat transfer rate \(q\) from the annular fin with stepped rectangular profile to the neighboring fluid can be calculated by applying Fourier’s law of heat conduction at the base:

\[
q = -kA \frac{dT(r_1)}{dr} = -(4\pi r_1 \delta k) \frac{dT(r_1)}{dr}
\]

(8)

whose dimensionless form is converted to

\[
Q = \frac{q}{4\pi r_1 k(T_b - T_f)}
\]

(9)

Performing the differentiation of the temperature for the thick part \(\frac{dT(r_1)}{dr}\) and later doing the algebra, the resulting expression for \(Q\) is

\[
Q = \xi R_1 Z_1 \left\{ \frac{I_1(ZR_1)K_0(ZR_1)+I_0(ZR_2)K_1(ZR_1)-A[I_0(ZR_1)K_1(ZR_1)+I_1(ZR_1)K_0(ZR_1)]}{I_0(ZR_2)K_0(ZR_1)-I_0(ZR_1)K_0(ZR_2)} \right\}
\]

(10)

where \(A\) is taken from eq. (6a).

The fin efficiency or dimensionless heat transfer rate was defined by Gardner (1945) as the ratio between the actual heat transfer rate \(Q\) and the ideal heat transfer rate \(Q_{ideal}\):

\[
\eta = \frac{Q}{Q_{ideal}}
\]

(11)

where the dimensionless ideal heat transfer rate \(Q_{ideal}\) requires that the entire fin is maintained at the base temperature \(T_b\),

\[
Q_{ideal} = \frac{q_{ideal}}{4\pi r_1 k(T_b - T_f)} = \frac{Bi}{2} \left( \frac{1}{R^2} - 1 \right)
\]

(12)

Introducing eqs. (10) and (12) into eq. (11) produces the working relation

\[
\eta = 2\xi R_1 Z_1 \frac{R^3}{Bi} \times \left\{ \frac{[I_1(ZR_1)K_0(ZR_1)+I_0(ZR_2)K_1(ZR_1)-A[I_0(ZR_1)K_1(ZR_1)+I_1(ZR_1)K_0(ZR_1)]}{[1-R^2+2R^2\xi(1-\delta)][I_0(ZR_2)K_0(ZR_1)-I_0(ZR_1)K_0(ZR_2)]} \right\}
\]

(13)

where \(A\) is taken from eq. (6a).
5. Results and Discussion

Before embarking on lengthy calculations for the fin efficiency and the tip temperature in annular fins with stepped profile evaluating equations (13) and (7), a first order engineering approximation may be done by computing the upper and lower bounds for the fin efficiency and the tip temperature in one annular fin of uniform thick thickness $\delta_1$ and in another annular fin of uniform thin thickness $\delta_2$. To this end, concise correlations equations for the calculation of the fin efficiency and the tip temperature in an annular fin of uniform thickness were developed by Campo and Stuffle (1997). In addition, with this information, mean fin efficiencies and mean tip temperatures could be determined and used as an adequate guideline.

As already mentioned, an annular fin of stepped rectangular profile is characterized by the four geometric parameters: 1) the dimensionless inner radius $R_1$, 2) the dimensionless step radius $R_2$, 3) the thickness ratio $\delta$, 4) the aspect ratio for thick part $\xi$ and one thermo-geometric parameter, the Biot number based on the inner radius $Bi_{h}$.

The two temperature distributions $\Theta(R)$ and $\phi(R)$ in eqs. (5a) and (5b), the tip temperature $\phi_{tip}=\phi(1)$ in eq. (7) and the fin efficiency $\eta$ in eq. (13) are evaluated numerically with the symbolic computer code Mathematica (www.wolfram.com/mathematica).

To comply with the quasi one-dimensionality criterion recommended by Lau and Tan (1973), we choose $Bi_{h}=h\delta/k=0.1$ and an aspect ratio for the thick part, $\xi=\delta/\delta_1=0.63$. This creates the ratio $Bi_{h1}/\xi=0.16$

Conversely, there is a relationship between the Biot number based on the inner radius $Bi_{h}$ and the Biot number based on the semi-thickness $Bi_{h1}$. The sequential steps are

$$Bi_{h}=(h\delta/k)(\delta_1/\delta)=(h\delta/k)(\delta_1/\delta_1)=(Bi_{h1}/\xi)$$

From here, $Bi_{h}=0.16$ and the dimensionless parameter is $Z_0=\sqrt{Bi_{h}/\xi}=0.50$.

Figure 2 displays the temperature distribution for an annular fin of stepped rectangular profile having $R_1=0.5$, $R_2=0.75$, $\delta=0.5$ along with $Z_0=0.5$. It is observed that the temperature matching at the border between the thick and thin parts $\Theta(0.75)$ and $\phi(0.75)$ is perfect.
Fig. 2. Temperature distributions in an annular fin with stepped rectangular profile having \( R_1 = 0.5, R_2 = 0.75, \delta = 0.5 \) and two \( Z_0 = 0.5, 0.7 \).

Table 1 Dimensionless tip temperature \( \phi(1) \) and fin efficiency \( \eta \) of an annular fin with stepped rectangular profile characterized by \( R_1 = 0.5, R_2 = 0.75, \delta = 0.5 \) with variable \( Z_0 \).

<table>
<thead>
<tr>
<th>( Z_0 )</th>
<th>( \phi(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.9923</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9698</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9342</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8878</td>
</tr>
<tr>
<td><strong>0.5</strong></td>
<td><strong>0.8335</strong></td>
</tr>
<tr>
<td>0.6</td>
<td>0.7742</td>
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<td><strong>0.7</strong></td>
<td><strong>0.7125</strong></td>
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<tr>
<td>0.8</td>
<td>0.6505</td>
</tr>
<tr>
<td>0.9</td>
<td>0.5902</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5325</td>
</tr>
</tbody>
</table>
The role played by $Z_0$ on the temperature is also shown in Figure 2. The implication of increasing $Z_0$ to 0.7 on the composite temperature distribution is caused either an increment in the mean convective heat transfer coefficient $h$ over the fin surface or a decrement in the thermal conductivity of the fin material $k$. Thereby, due to the elevation in $Z_0$, the temperatures are pushed down sharply.

Additionally, Table 1 lists the dimensionless tip temperature $\phi(1)$ and fin efficiency $\eta$ of an annular fin with stepped rectangular consisting in $R_1 = 0.5$, $R_2 = 0.75$, $\delta = 0.5$ associated with Figure 2, but now varying $Z_0$ between zero and one.

### Funding Source

None

### Conflict of Interest

None

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