The Stability Analysis of a Three Species Syn-Eco-System with Mortality Rates

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Abstract

In this paper, the system comprises of a commensal (S1) common to two hosts S2 and S3 with mortality rate for all the three species. The model equations constitute a set of three first order non-linear simultaneous differential equations. Criteria for the asymptotic stability of all the eight equilibrium states are established. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. Trajectories of the perturbations over the equilibrium states are illustrated. Further the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order scheme.

Keywords: Commensal; Equilibrium State; Host; Trajectories; Stable; Unstable

2010 Mathematics Subject Classification: 92D25; 92D40

1. Introduction

Ecology, a branch evolutionary biology, deals with living species that coexist in a physical environment and sustain themselves on common resources. It is a common observation that the species of same nature cannot flourish in isolation without any interaction with species of different kinds. Syn-ecology is an ecosystem comprising of two or more distinct species. Species interact with each other in one way or other. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation Parasitism and so on. Lotka (1925) and Volterra (1931) pioneered theoretical ecology significantly and opened new eras in the field of life and biological sciences.

Mathematical Modeling is a vital role in providing insight in to the mutual relationships between the interacting species. The general concepts of modeling have been discussed by several authors

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The present investigation is on an analytical study of three species \((S_1, S_2, S_3)\) syn-eco system with mortality rate for all the three species. The system comprises of a commensal \((S_1)\), two hosts \(S_2\) and \(S_3\) ie, \(S_2\) and \(S_3\) both benefit \(S_1\), without getting themselves affected either positively or adversely. Further \(S_2\) is a commensal of \(S_3\) and \(S_3\) is a host of both \(S_1, S_2\).

**Commensalism** is a symbiotic interaction between two populations where one population \((S_1)\) gets benefit from \((S_2)\) while the other \((S_2)\) is neither harmed nor benefited due to the interaction with \((S_1)\). The benefitted species \((S_1)\) is called the commensal and the other \((S_2)\) is called the host. A real-life example of commensalism is a flatworm attached to the horse crab and eating the crab’s food, while the crab is not put to any disadvantage.

### 2. Notation Adopted

- \(N_i(t)\) : The population strength of \(S_i\) at time \(t\), \(i = 1,2,3\)
- \(t\) : Time instant
- \(d_i\) : Natural death rate of \(S_i\), \(i = 1,2,3\)
- \(a_{ii}\) : Self inhibition coefficient of \(S_i\), \(i = 1,2,3\)
- \(a_{12}, a_{13}\) : Interaction coefficients of \(S_1\) due to \(S_2\) and \(S_1\) due to \(S_3\)
- \(a_{23}\) : Interaction coefficient of \(S_2\) due to \(S_3\)
- \(e_i = \frac{d_i}{a_{ii}}\) : Extinction coefficient of \(S_i\), \(i = 1,2,3\)

Further the variables \(N_1, N_2, N_3\) are non-negative and the model parameters \(d_1, d_2, d_3, a_{11}, a_{12}, a_{22}, a_{33}, a_{13}, a_{23}, e_1, e_2, e_3\) are assumed to be non-negative constants.

### 3. Basic Equations of the Model

The model equations for the three species syn ecosystem are given by the following system of first order non-linear ordinary differential equations.
4. Equilibrium States

The system under investigation has eight equilibrium states given by $\frac{dN_i}{dt} = 0$, $i = 1, 2, 3$

**Fully washed out state.**

$E_1 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = 0$

**States in which only one of the tree species is survives while the other two are not.**

$E_2 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = -e_3$

$E_3 : \overline{N}_1 = 0, \overline{N}_2 = -e_2, \overline{N}_3 = 0$

$E_4 : \overline{N}_1 = -e_1, \overline{N}_2 = 0, \overline{N}_3 = 0$

**States in which only two of the tree species are survives while the other one is not.**

$E_5 : \overline{N}_1 = 0, \overline{N}_2 = -\left( e_2 + \frac{a_{23}e_3}{a_{22}} \right), \overline{N}_3 = -e_3$

$E_6 : \overline{N}_1 = -\left( e_1 + \frac{a_{13}e_3}{a_{11}} \right), \overline{N}_2 = 0, \overline{N}_3 = -e_3$
$E_7: \overline{N}_1 = -\left( e_1 + \frac{a_{23}e_2}{a_{11}} \right), \overline{N}_2 = -e_2, \overline{N}_3 = 0$

The co-existent state (or) normal steady state.

$E_8: \overline{N}_1 = -\frac{1}{a_{11}} \left[ d_1 + \frac{a_{12}}{a_{22}} \left( d_2 + a_{23}e_3 \right) + a_{13}e_3 \right], \overline{N}_2 = -\left( e_2 + \frac{a_{23}e_3}{a_{22}} \right), \overline{N}_3 = -e_3$

5. Stability of the Equilibrium States

Let $N = (N_1, N_2, N_3) = \overline{N} + U$

where $U = (u_1, u_2, u_3)^T$ is a small perturbation over the equilibrium state $\overline{N} = (\overline{N}_1, \overline{N}_2, \overline{N}_3)$.

The basic equations (1), (2) and (3) are quasi-linearized to obtain the equations for the perturbed state as,

$$\frac{dU}{dt} = AU \tag{5}$$

with

$$A = \begin{bmatrix}
-d_1 - 2a_{11}\overline{N}_1 + a_{12}\overline{N}_2 + a_{13}\overline{N}_3 & a_{12}\overline{N}_2 & a_{13}\overline{N}_3 \\
0 & -d_2 - 2a_{22}\overline{N}_2 + a_{23}\overline{N}_3 & a_{23}\overline{N}_3 \\
0 & 0 & -d_3 - 2a_{33}\overline{N}_3
\end{bmatrix} \tag{6}$$

The characteristic equation for the system is $\det [A - \lambda I] = 0 \tag{7}$

The equilibrium state is stable, if all the roots of the equation (7) are negative in case they are real or have negative real parts, in case they are complex.

5.1 Fully washed out state

In this case, we get

$$A = \begin{bmatrix}
-d_1 & 0 & 0 \\
0 & -d_2 & 0 \\
0 & 0 & -d_3
\end{bmatrix} \tag{8}$$
The characteristic equation is 
\[(\lambda + d_1)(\lambda + d_2)(\lambda + d_3) = 0\]  \hspace{1cm} (9)

The characteristic roots of (9) are \(-d_1, -d_2, \) and \(-d_3\). Since all the three roots are negative. Hence the fully washed out state is **stable** and the solutions of the equations (5) are

\[u_i = u_{i0} e^{-d_it}, \quad i = 1, 2, 3\]  \hspace{1cm} (10)

where \(u_{i0}, u_{20}, u_{30}\) are the initial values of \(u_1, u_2, u_3\) respectively.

**Trajectories of perturbations**

The trajectories in \(u_1 - u_2\) and \(u_2 - u_3\) planes are

\[
\begin{pmatrix}
\frac{u_1}{u_{10}} \\
\frac{u_2}{u_{20}} \\
\frac{u_3}{u_{30}}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{d_1} \\
\frac{1}{d_2} \\
\frac{1}{d_3}
\end{pmatrix}
\]

\[5.2 \text{ Equilibrium state} \quad E_2: \tilde{N}_1 = 0, \tilde{N}_2 = 0, \tilde{N}_3 = -e_3\]

In this case, we have

\[A = \begin{bmatrix}
-(d_1 + a_{13}e_3) & 0 & 0 \\
0 & -(d_2 + a_{23}e_3) & 0 \\
0 & 0 & d_3
\end{bmatrix}\]  \hspace{1cm} (11)

The characteristic roots are \(-(d_1 + a_{13}e_3), -(d_2 + a_{23}e_3)\) and \(d_3\). Since one of the three roots is positive, hence the state is **unstable** and (5) yield the solution curves.

\[u_1 = u_{10} e^{-(d_1 + a_{13}e_3)t}; \quad u_2 = u_{20} e^{-(d_2 + a_{23}e_3)t}; \quad u_3 = u_{30} e^{d_3t}\]  \hspace{1cm} (12)

**Trajectories of perturbations**

The trajectories in the \(u_1 - u_2\) and \(u_2 - u_3\) planes are given by

\[
\begin{pmatrix}
\frac{u_1}{u_{10}} \\
\frac{u_2}{u_{20}} \\
\frac{u_3}{u_{30}}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{d_1 + a_{13}k_3} \\
\frac{1}{d_2 + a_{23}k_3} \\
\frac{1}{d_3}
\end{pmatrix}
\]
5.3 Equilibrium state $E_3: \overline{N}_1 = 0, \overline{N}_2 = -e_2, \overline{N}_3 = 0$

In this case, (6) becomes

$$A = \begin{bmatrix} -(d_1 + a_{12}e_2) & 0 & 0 \\ 0 & d_2 & -a_{23}e_2 \\ 0 & 0 & -d_3 \end{bmatrix}$$

(13)

The characteristic roots are $-(d_1 + a_{12}e_2), d_2$, and $-d_3$. Since one of the three roots is positive, hence the state is unstable and (5) yield the solution curves.

$$u_1 = u_{10} e^{-(d_1 + a_{12}e_2)t} ; u_2 = u_{20} - \frac{a_{23}e_2u_{30}}{d_2 + d_3} e^{d_2t} + \frac{a_{23}e_2u_{30}}{d_2 + d_3} u_{30} e^{-d_3t} ; u_3 = u_{30} e^{-d_3t}$$

(14)

Trajectories of perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_2 = \left( u_{20} - \frac{a_{23}e_2u_{30}}{d_2 + d_3} \right) \frac{d_3}{d_1 + a_{12}e_2} \left( \frac{u_1}{u_{10}} \right) \frac{d_2}{d_2 + d_3} + \frac{a_{23}e_2u_{30}}{d_2 + d_3} \left( \frac{u_3}{u_{10}} \right) \frac{d_3}{d_1 + a_{12}e_2}$$

and

$$u_2 = \left( u_{20} - \frac{a_{23}e_2u_{30}}{d_2 + d_3} \right) \frac{d_2}{d_1 + a_{12}e_2} \left( \frac{u_1}{u_{10}} \right) \frac{d_2}{d_2 + d_3} + \frac{a_{23}e_2u_{30}}{d_2 + d_3}$$

5.4 Equilibrium state $E_4: \overline{N}_1 = -e_1, \overline{N}_2 = 0, \overline{N}_3 = 0$

In this case, (6) becomes

$$A = \begin{bmatrix} d_1 & -a_{12}e_1 & -a_{13}e_1 \\ 0 & -d_2 & 0 \\ 0 & 0 & -d_3 \end{bmatrix}$$

(15)

The characteristic roots are $d_1, -d_2$ and $-d_3$. Since one of the three roots is positive, hence the state is unstable and the equations (5) yield the solutions.
\[ u_1 = \left( u_{10} - \frac{a_{12}e_1u_{20}}{d_1 + d_2} - \frac{a_{13}e_1u_{30}}{d_1 + d_3} \right) e^{d_1t} + \frac{a_{12}e_1}{d_1 + d_2} u_{20} e^{-d_2t} + \frac{a_{13}e_1}{d_1 + d_3} u_{30} e^{-d_3t}, \]
\[ u_2 = u_{20} e^{-d_2t} \text{ and } u_3 = u_{30} e^{-d_3t} \]

**Trajectories of perturbations**

The trajectories in the \( u_1 - u_2 \) and \( u_2 - u_3 \) planes are given by

\[ u_1 = \left( u_{20} - \frac{a_{12}e_1u_{20}}{d_1 + d_2} - \frac{a_{13}e_1u_{30}}{d_1 + d_3} \right) \frac{u_2}{u_{20}} \frac{d_1}{d_2} + \frac{a_{12}e_1}{d_1 + d_2} \frac{u_2}{u_{20}} + \frac{a_{13}e_1}{d_1 + d_3} \frac{u_2}{u_{20}} \frac{d_1}{d_3}; \left( \frac{u_2}{u_{20}} \right)^{d_1} = \left( \frac{u_3}{u_{30}} \right)^{d_2} \]

**5.5 Equilibrium state** \( E_s : \vec{N}_1 = 0, \vec{N}_2 = -\left( e_2 + \frac{a_{23}e_3}{a_{22}} \right), \vec{N}_3 = -e_3 \)

In this case, we get

\[
A = \begin{bmatrix}
-\varphi_1 & 0 & 0 \\
0 & d_2 + a_{23}e_3 & -\frac{a_{23}e_3}{a_{22}} (d_2 + a_{23}e_3) \\
0 & 0 & d_3
\end{bmatrix}
\]

where,

\[
\varphi_1 = \left[ d_1 + a_{13}e_3 + \frac{a_{12}}{a_{22}} (d_2 + a_{23}e_3) \right] > 0
\]

The characteristic roots are \(-\varphi_1, d_2 + a_{23}e_3\) and \(d_3\). Since two of the three roots are positive, hence the state is **unstable** and (5) yield the solution curves.

\[ u_1 = u_{10} e^{-\varphi_1 t}; u_2 = (u_{20} - \xi u_{30}) e^{(d_2 + a_{23}e_3)t} + \xi u_{30} e^{d_3t}; u_3 = u_{30} e^{d_3t} \]

where,

\[
\xi = \frac{a_{23} (d_2 + a_{23}e_3)}{a_{22} (d_2 + a_{23}e_3 - d_3)}
\]

with,

\[ d_2 + a_{23}e_3 \neq d_3 \]
Trajectories of perturbations

The trajectories in the \( u_1 - u_2 \) and \( u_2 - u_3 \) planes are given by

\[
\begin{align*}
u_2 &= (u_{20} - \xi u_{30}) \left( \frac{u_1}{u_{10}} \right)^{\frac{(d_{12} + a_{12} e_3)}{a_{11}}} + \xi u_{30} \left( \frac{u_1}{u_{10}} \right)^{\frac{d_1}{a_{11}}} ; \quad u_2 &= (u_{20} - \xi u_{30}) \left( \frac{u_3}{u_{30}} \right)^{\frac{a_{13} e_3 + d_3}{d_3}} + \xi u_3
\end{align*}
\]

5.6 Equilibrium state \( E_0 : \vec{N}_1 = - \left( e_1 + \frac{a_{13} e_3}{a_{11}} \right), \vec{N}_2 = 0, \vec{N}_3 = -e_3 \)

In this case, we have

\[
A = \begin{bmatrix}
  d_1 + a_{13} e_3 & -\frac{a_{12}}{a_{11}} (d_1 + a_{13} e_3) & -\frac{a_{13}}{a_{11}} (d_1 + a_{13} e_3) \\
  0 & -(d_2 + a_{23} e_3) & 0 \\
  0 & 0 & d_3
\end{bmatrix}
\]

The characteristic roots are \( d_1 + a_{13} e_3, -(d_2 + a_{23} e_3) \) and \( d_3 \). Since two of the three roots are positive, hence the state is **unstable** and (5) yield the solution curves.

\[
\begin{align*}
u_1 &= (u_{10} - \tilde{\xi}_1 u_{20} - \tilde{\xi}_2 u_{30}) e^{(d_1 + a_{13} e_3)t} + \tilde{\xi}_1 u_{20} e^{-(d_2 + a_{23} e_3)t} + \tilde{\xi}_2 u_{30} e^{d_3 t}, \\
u_2 &= u_{20} e^{-(d_2 + a_{23} e_3)t} \quad \text{and} \quad u_3 = u_{30} e^{d_3 t}
\end{align*}
\]

where,

\[
\tilde{\xi}_1 = \frac{a_{12} (d_1 + a_{13} e_3)}{a_{11} \left[ d_1 + d_2 + e_3 (a_{13} + a_{23}) \right]} \quad \text{and} \quad \tilde{\xi}_2 = \frac{a_{13} (d_1 + a_{13} e_3)}{a_{11} \left[ d_1 + a_{13} e_3 - d_3 \right]}
\]

with,

\[
d_1 + a_{13} e_3 \neq d_3
\]

Trajectories of perturbations

The trajectories in the \( u_1 - u_2 \) and \( u_2 - u_3 \) planes are
\[ u_1 = \left( u_{10} - \frac{\xi_1}{\tau_1} u_{20} - \frac{\xi_2}{\tau_2} u_{30} \right) \left( \frac{d_1 + a_{12} e_2}{d_1 + d_2 + a_{12} e_2} \right) u_2 + \frac{\xi_1}{\tau_1} u_{20} + \frac{\xi_2}{\tau_2} u_{30} \left( \frac{d_1}{d_1 + d_2 + a_{12} e_2} \right) \] 

and

\[ \left( \frac{u_2}{u_{20}} \right)^{d_1_1} = \left( \frac{u_3}{u_{30}} \right)^{-(d_2 + a_{12} e_2)} \]

5.7 Equilibrium state \( E_7 : \bar{N}_1 = -\left( e_1 + \frac{a_{12} e_2}{a_{11}} \right), \bar{N}_2 = -e_2, \bar{N}_3 = 0 \)

In this case, (6) becomes

\[
A = \begin{bmatrix}
d_1 + a_{12} e_2 - \frac{a_{12}}{a_{11}} (d_1 + a_{12} e_2) & -\frac{a_{13}}{a_{11}} (d_1 + a_{12} e_2) \\
0 & d_2 & -a_{13} e_2 \\
0 & 0 & -d_3
\end{bmatrix}
\]

The characteristic roots are \( d_1 + a_{12} e_2, d_2 \) and \(-d_3\). Since two of the three roots are positive, hence the state is \textbf{unstable} and (5) yield the solution curves.

\[
u_1 = \left( u_{10} - \frac{\xi_3}{\tau_3} - \frac{\xi_4}{\tau_4} \right) e^{(d_1 + a_{12} e_2) t} + \frac{\xi_3}{\tau_3} e^{d_2 t} + \frac{\xi_4}{\tau_4} e^{d_3 t},
\]

\[
u_2 = \left( u_{20} - \frac{\xi_5}{\tau_5} \right) e^{d_2 t} + \frac{\xi_5}{\tau_5} e^{-d_2 t} \quad \text{and} \quad \nu_3 = u_{30} e^{-d_2 t}
\]

where,

\[
\xi_3 = \frac{a_{12} (d_1 + a_{12} e_2) (\xi_3 u_{20})}{a_{11} (d_2 - d_1 - a_{12} e_2)}; \xi_4 = \frac{(d_1 + a_{12} e_2) (a_{12} \xi_3 + a_{13} u_{30})}{a_{11} (d_1 + d_3 + a_{12} e_2)}; \xi_5 = \frac{a_{12} e_2 u_{30}}{d_2 + d_3}
\]

with,

\[ d_2 \neq d_1 + a_{12} e_2 \]

Trajectories of perturbations

The trajectories in \( u_1 - u_3 \) and \( u_2 - u_3 \) planes are given by
The characteristic roots are $\varphi_1$, $d_2 + a_{23}e_3$ and $d_3$, and all these roots are positive, hence the state is \textbf{unstable} and the equations (5) yield the solutions.

$$u_1 = (u_{10} - \varphi_2 - \varphi_3) e^{\varphi_1 t} + \varphi_2 e^{(d_2 + a_{23}e_3)t} + \varphi_3 e^{d_3t},$$

$$u_2 = (u_{20} - \varphi_4) e^{(d_2 + a_{23}e_3)t} + \varphi_4 e^{d_3t} \quad \text{and} \quad u_3 = u_{30} e^{d_3t}$$  \begin{align}  \label{28} &u_1 = (u_{10} - \varphi_2 - \varphi_3) \left(\frac{u_3}{u_{30}}\right)^d + \varphi_2 \left(\frac{u_3}{u_{30}}\right)^d \frac{a_{23}e_3 + d_3}{d_3} + \varphi_3 \left(\frac{u_3}{u_{30}}\right)^d \frac{d_3}{d_3} + \varphi_4 \frac{u_3}{u_{30}} \end{align}

where,

$$\varphi_2 = \frac{a_{12} \varphi_1 (\varphi_4 - u_{20})}{a_{11} (d_2 + a_{23}e_3 - \varphi_1)}; \varphi_3 = \frac{\varphi_1 (a_{12} \varphi_4 + a_{13} u_{30})}{a_{11} (\varphi_1 - d_3)}; \varphi_4 = \frac{a_{23} (d_2 + a_{23}e_3)}{a_{22} (d_2 + a_{23}e_3 - d_3)}$$  \begin{align}  \label{29} &\varphi_2 = -a_{12} \varphi_1 (\varphi_4 - u_{20}); \varphi_3 = \varphi_1 (a_{12} \varphi_4 + a_{13} u_{30}); \varphi_4 = \frac{a_{23} (d_2 + a_{23}e_3)}{a_{22} (d_2 + a_{23}e_3 - d_3)} \end{align}

with,

$$d_2 + a_{23}e_3 \neq \varphi_1; \varphi_1 \neq d_3 \quad \text{and} \quad d_2 + a_{23}e_3 \neq d_3$$

\textbf{Trajectories of perturbations}

Trajectories in $u_1 - u_3$ and $u_2 - u_1$ planes are given by

$$u_1 = (u_{10} - \varphi_2 - \varphi_3) \left(\frac{u_3}{u_{30}}\right)^d + \varphi_2 \left(\frac{u_3}{u_{30}}\right)^d \frac{a_{23}e_3 + d_3}{d_3} + \varphi_3 \left(\frac{u_3}{u_{30}}\right)^d \frac{d_3}{d_3} + \varphi_4 \frac{u_3}{u_{30}}$$

$$u_2 = (u_{20} - \varphi_4) e^{(d_2 + a_{23}e_3)t} + \varphi_4 e^{d_3t} \quad \text{and} \quad u_3 = u_{30} e^{d_3t}$$
6. A Numerical Approach of the Growth Rate Equations

The numerical solutions of the growth rate equations (1), (2) and (3) computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. The results are illustrated in Figures 1 to 4.

Fig. 1. Variation of population against time for $d_1=1.84$, $d_2=4.24$, $d_3=0.516$, $a_{11}=3.6$, $a_{22}=9.65$, $a_{33}=1.4$, $a_{12}=0.6$, $a_{13}=12.8$, $a_{23}=24.35$, $N_{10}=12.96$, $N_{20}=10.36$, $N_{30}=4.36$

Fig. 2. Variation of population against time for $d_1=0.28$, $d_2=0.09$, $d_3=1.7$, $a_{11}=0.468$, $a_{22}=1.782$, $a_{33}=11.52$, $a_{12}=2.286$, $a_{13}=1.35$, $a_{23}=2.304$, $N_{10}=0.432$, $N_{20}=5.094$, $N_{30}=1.872
Fig. 3. Variation of population against time for $d_1=3.01$, $d_2=0.342$, $d_3=0.432$, $a_{11}=1.278$, $a_{22}=3.978$, $a_{33}=0.432$, $a_{12}=0.468$, $a_{13}=0.63$, $a_{23}=1.224$, $N_{10}=2.97$, $N_{20}=1.746$, $N_{30}=5.004$

Fig. 4. Variation of population against time for $d_1=0.1$, $d_2=3.852$, $d_3=2.448$, $a_{11}=2.664$, $a_{22}=0.036$, $a_{33}=0.432$, $a_{12}=0.594$, $a_{13}=1.656$, $a_{23}=5.292$, $N_{10}=0.306$, $N_{20}=3.402$, $N_{30}=2.142$

7. Observations of the above Graphs

Case 1: In this case the third species dominates over the other two throughout. The third species has the least natural death rate. This is a situation at the self inhibition coefficient of the second species is highest. Further it is evident that all the three species asymptotically converge to the equilibrium point as illustrated in Figure 1.

Case 2: In this case the third species has the highest natural death rate. Initially the second and third species dominates over the first till the time instant $t^* = 0.08$ and $t^* = 0.24$ respectively and
thereafter the dominance is reversed. Further we notice that the first species has the least self inhibition coefficient. This is shown in Figure 2.

Case 3: In this case the natural death rates of the second and third species are almost equal. Initially the first species dominates over the second till the time instant \( t^* = 0.52 \) and thereafter the dominance is reversed. It is evident that all the three species asymptotically converge to the equilibrium point. This is illustrated in Figure 3.

Case 4: In this case the first species has the least initial value. The second and third species dominates over the first initially up to the time \( t^* = 1.38 \) and \( t^* = 0.25 \) after which the dominance is reversed. Further it is evident that all the three species asymptotically converge to the equilibrium point (Figure 4).

8. Conclusion

The present paper deals with an investigation on the stability of a three species syn eco-system with mortality rate for all the three species. The system comprises of a commensal \( (S_1) \), two hosts \( S_2 \) and \( S_3 \) i.e., \( S_2 \) and \( S_3 \) both benefit \( S_1 \), without getting themselves effected either positively or adversely. In this paper we established all possible equilibrium states. It is conclude that, in all eight equilibrium states, only the fully washed out state is **stable** and rest of them are **unstable**. Further the growth rates of the species are numerically estimated using Runge-Kutta fourth order method.

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References


