Columbia International Publishing
Contemporary Mathematics and Statistics
(2015) Vol. 3 No. 1 pp. 8-27
doi:10.7726/cms.2015.1002
Research Article

Filtering Problem for Random Processes with Stationary Increments

Maksym Luz[1], and Mikhail Moklyachuk[1*]

Received 2 December 2014; Published online 17 October 2015

© The author(s) 2014. Published with open access at www.uscip.us

Abstract

The problem of mean-square optimal filtering is considered for the linear functionals $A\xi = \int_0^{\infty} a(t)\xi(-t)dt$ and $A_T\xi = \int_0^{T} a(t)\xi(-t)dt$ which depend on the unknown values of a random process $\xi(t)$ with stationary n th increments from observations of the process $\xi(t)+\eta(t)$ at points $t \le 0$, where $\eta(t)$ is an uncorrelated with $\xi(t)$ stationary process. Formulas for calculation values of the mean-square errors and the spectral characteristics of the optimal linear estimates of the functionals are derived under the condition of spectral certainty where the spectral densities of the processes are exactly known. In the case of spectral uncertainty where the spectral densities of the processes are not exactly known, but a class of admissible spectral densities is given, relations that determine the least favorable spectral densities and the minimax spectral characteristics are proposed.

Keywords: Random Process with n th Stationary Increments; Minimax-Robust Estimate; Mean-Square Error; Least Favorable Spectral Density; Minimax Sspectral Characteristic.

2000 Mathematics Subject Classification: Primary: 60G10, 60G25, 60G35, Secondary: 62M20, 93E10, 93E11

1. Introduction

The problems of estimating the unobserved values of random processes play an important role in both theoretical and applied probability. Effective methods of extrapolation, interpolation and filtering of stationary random processes and sequences were developed by Kolmogorov (see selected works by Kolmogorov, 1992), Wiener (see the book by Wiener, 1966) and Yaglom (see

*Corresponding e-mail: Moklyachuk@gmail.com
1 Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine

books by Yaglom, 1987a; 1987b). Since that time a lot of generalizations of stationary processes have been introduced, some of which are discussed, for example, in books by Yaglom (1987a; 1987b). One important generalization was proposed in a paper by Yaglom (1955), in which he considered random processes with stationary n th increments. He described the spectral representation of the stationary increments and the canonical factorization of the spectral density, solved extrapolation problem for unknown value of random process with stationary increments and discussed a few examples. Further investigations of such random processes were made in the articles by Yaglom and Pinsker (Yaglom and Pinsker, 1954; Pinsker, 1955; Yaglom, 1957).

The crucial assumption of most of the papers dedicated to the problem of estimating the unobserved values of random processes is that the spectral densities of the involved random processes are exactly known. However, the developed methodscan not be directly applied to practical estimation problems, because complete information of the spectral densities is impossible in most cases. In such situations one finds parametric or nonparametric estimates of the unknown spectral densities. Then the classical estimation methods are applied under the assumption that the estimated densities are true. This procedure can result in significant increasing of the value of the error of estimate as Vastola and Poor (1983) have demonstrated with the help of some examples. This is a reason to derive estimates which would be optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximum of the mean-square errors for all spectral densities from the set of admissible spectral densities simultaneously. A survey of results in minimax (robust) methods of data processing till 1984 can be found in the paper by Kassam and Poor (1985). Note, that Grenander (1957) was the first one who proposed this method for solving the extrapolation problem for stationary processes. Later Franke and Poor (Franke and Poor, 1984; Franke, 1985) applied the convex optimization methods to investigate the minimax-robust extrapolation and interpolation problems. In papers by Moklyachuk (1989 – 2008a) the minimax-robust extrapolation, interpolation and filtering problems are investigated for stationary processes. The papers by Moklyachuk and Masyutka (2006 – 2012) are dedicated to minimax-robust extrapolation, interpolation and filtering problems for vector-valued stationary processes and sequences. Dubovetska et al. (2012) solved the problem of minimax-robust interpolation for another generalization of stationary processes – periodically correlated sequences. In the further papers Dubovetska and Moklyachuk (2013 – 2014) investigated the minimax-robust extrapolation, interpolation and filtering problems for periodically correlated processes.

The minimax-robust extrapolation, interpolation and filtering problems for stochastic sequences and random processes with n th stationary increments are investigated by Luz and Moklyachuk (Luz and Moklyachuk, 2012 – 2015b; Moklyachuk and Luz, 2013). In particular, the minimax-robust filtering problem for such sequences is investigated in papers by Luz and Moklyachuk (2013b; 2014b). Random processes with stationary increments with continuous time are considered in the articles by Luz and Moklyachuk (2014a; 2015a), where the authors investigated the extrapolation and the interpolation problems.

In this article we propose solutions to the problem of the mean-square optimal estimation of the linear functionals $A_T\xi = \int_0^T a(t)\xi(-t)dt$ and $A\xi = \int_0^\infty a(t)\xi(-t)dt$ which are determined by the unknown values of a random process $\xi(t)$ with stationary n-th increments from observations of a process $\xi(t)+\eta(t)$ at points $t \le 0$, where $\eta(t)$ is an uncorrelated with $\xi(t)$ stationary process, under the condition of spectral certainty as well as under the condition of spectral uncertainty. Formulas for calculation values of the mean-square errors and the spectral characteristics of the optimal linear estimates of the functionals are derived under the condition of spectral certainty where the spectral densities of the processes are exactly known. In the case of spectral uncertainty where the spectral densities of the processes are not exactly known, but a class of admissible spectral densities is given, relations that determine the least favorable spectral densities and the minimax spectral characteristics are derived for some wide classes of spectral densities.

2. Stationary Random Increment Process: Spectral Representation

In this section we present a brief review of the spectral theory of random processes with stationary increment developed by Yaglom (1955).

Definition 2.1 *For a given random process* $\{\xi(t), t \in \mathbb{R}\}$ *the process*

$$\xi^{(n)}(t,\tau) = (1-B_\tau)^n \xi(t) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \xi(t - l\tau), \qquad (1)$$

where B_τ is a backward shift operator with step $\tau \in \mathbb{R}$, such that $B_\tau \xi(t) = \xi(t-\tau)$, is called the random nth increment with step $\tau \in \mathbb{R}$ generated by the random process $\xi(t)$.

The random nth increment process $\xi^{(n)}(t,\tau)$ satisfies relations:

$$\xi^{(n)}(t,-\tau) = (-1)^n \xi^{(n)}(t + n\tau, \tau), \qquad (2)$$

$$\xi^{(n)}(t,k\tau) = \sum\nolimits_{l=0}^{(k-1)n} A_l \xi^{(n)}(t - l\tau, \tau), \quad \forall k \in \mathbb{N}, \qquad (3)$$

where coefficients $\{A_l, l = 0,1,2,\dots,(k-1)n\}$ are taken from the representation

$$\left(1 + x + \dots + x^{k-1}\right)^n = \sum_{l=0}^{(k-1)n} A_l x^l.$$

Definition 2.2 *The random* n *th increment process* $\xi^{(n)}(t,\tau)$ *generated by a random process* $\{\xi(t), t \in R\}$ *is wide sense stationary if the mathematical expectations*

$$\mathrm{E}\xi^{(n)}(t_0,\tau) = c^{(n)}(\tau),$$

$$\mathrm{E}\xi^{(n)}(t_0 + t, \tau_1)\overline{\xi^{(n)}(t_0,\tau_2)} = D^{(n)}(t,\tau_1,\tau_2)$$

exist for all $t_0, \tau, t, \tau_1, \tau_2$ *and do not depend on* t_0. *The function* $c^{(n)}(\tau)$ *is called the mean value of the* n *th increment and the function* $D^{(n)}(t,\tau_1,\tau_2)$ *is called the structural function of the stationary* n *th increment (or the structural function of* n *th order of the random process* $\{\xi(t), t \in \mathbb{R}\}$ *).*

The random process $\{\xi(t), t \in \mathbb{R}\}$ which determines the stationary n th increment process $\xi^{(n)}(t,\tau)$ by formula (1) is called the process with stationary n th increments.

The following theorem provides representations of the mean value and the structural function of the random stationary n th increment process $\xi^{(n)}(t,\tau)$.

Theorem 2.1 *The mean value $c^{(n)}(\tau)$ and the structural function $D^{(n)}(t,\tau_1,\tau_2)$ of the random stationary n th increment process $\xi^{(n)}(t,\tau)$ can be represented in the following forms*

$$c^{(n)}(\tau) = c\tau^n, \tag{4}$$

$$D^{(n)}(t,\tau_1,\tau_2) = \int_{-\infty}^{\infty} e^{i\lambda t}(1-e^{-i\tau_1\lambda})^n(1-e^{i\tau_2\lambda})^n \frac{(1+\lambda^2)^n}{\lambda^{2n}} dF(\lambda), \tag{5}$$

where c is a constant, $F(\lambda)$ is a left-continuous nondecreasing bounded function with $F(-\infty)=0$. The constant c and the function $F(\lambda)$ are determined uniquely by the increment process $\xi^{(n)}(t,\tau)$.

On the other hand, the function $c^{(n)}(\tau)$ which has form (4) with the constant c and the function $D^{(n)}(t,\tau_1,\tau_2)$ which has form (5) with the function $F(\lambda)$ which satisfies the indicated conditions are the mean value and the structural function of some stationary n th increment process $\xi^{(n)}(t,\tau)$.

From representation (5) of the structural function of the stationary n th increment process $\xi^{(n)}(t,\tau)$ and the Karhunen theorem (see Karhunen, 1947), we obtain the following spectral representation of the stationary increment process $\xi^{(n)}(t,\tau)$:

$$\xi^{(n)}(t,\tau) = \int_{-\infty}^{\infty} e^{it\lambda}(1-e^{-i\lambda\tau})^n \frac{(1+i\lambda)^n}{(i\lambda)^n} dZ_{\xi^{(n)}}(\lambda), \tag{6}$$

where $Z_{\xi^{(n)}}(\lambda)$ is a random process with independent increments on $\mathbb{R}$ connected with *the spectral function* $F(\lambda)$ from representation (5) by the relation

$$\mathsf{E}|Z_{\xi^{(n)}}(t_2) - Z_{\xi^{(n)}}(t_1)|^2 = F(t_2) - F(t_1) < \infty \quad \textit{for all } t_2 > t_1. \tag{7}$$

Without loss of generality we will consider the random increment process $\xi^{(n)}(t,\tau)$ with the step $\tau > 0$ and the mean value 0.

3. Filtering Problem

Consider a random process $\{\xi(t), t \in \mathbb{R}\}$ which defines a stationary n th increment process $\xi^{(n)}(t,\tau)$ with the absolutely continuous spectral function $F(\lambda)$ which has the spectral density

$f(\lambda)$. Assume that we observe another process $\zeta(t)=\xi(t)+\eta(t)$ on the time interval $t\le 0$. The filtering problem means that we need to restore values of the original process $\xi(t)$ at points $t\le 0$.

In this article we focus on finding the mean-square optimal estimates of the linear functionals $A_T\xi=\int_0^T a(t)\xi(-t)dt$ and $A\xi=\int_0^\infty a(t)\xi(-t)dt$ under the assumption that the noise process $\eta(t)$ is an uncorrelated with $\xi(t)$ stationary process with the spectral density $g(\lambda)$ and $\mathrm{E}\eta(t)=0$.

We will require that the spectral densities $f(\lambda)$ and $g(\lambda)$ of the random processes $\xi(t)$ and $\eta(t)$ satisfy the minimality conditions:

$$\int_{-\infty}^{\infty}\frac{|\gamma(\lambda)|^2}{f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda)}d\lambda<\infty,\quad \int_{-\infty}^{\infty}\frac{|\gamma(\lambda)|^2\,\lambda^{2n}}{|1-e^{i\lambda\tau}|^{2n}\,(1+\lambda^2)^n(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda))}d\lambda<\infty \tag{8}$$

for some non-zero function of the form $\gamma(\lambda)=\int_0^\infty\alpha(t)e^{-i\lambda t}dt$, $\int_0^\infty|\alpha(t)|^2\,dt<\infty$.

We will also require that the function $\mathbf{a}_\tau(t)$, $t\ge -\tau n$, being defined later satisfy the conditions

$$\int_0^\infty|\mathbf{a}_\tau(t-\tau n)|\,dt<\infty,\quad \int_0^\infty t\,|\mathbf{a}_\tau(t-\tau n)|^2\,dt<\infty. \tag{9}$$

These conditions guarantee correctness of the following constructions.

To solve the filtering problem for the functional $A\xi$ and the random processes $\xi(t)$ and $\eta(t)$ described above we represent the functional $A\xi=\int_0^\infty a(t)\xi(-t)dt$ in the form

$$A\xi=A\zeta-A\eta,$$

where $A\zeta=\int_0^\infty a(t)\zeta(-t)dt$, $A\eta=\int_0^\infty a(t)\eta(-t)dt$. We can find the exact value of the functional $A\zeta$ from observations of the process $\zeta(t)$ at points $t\le 0$. So it is sufficient to construct an estimate $\hat{A}\eta$ of the functional $A\eta$ in order to find the estimate $\hat{A}\xi$:

$$\hat{A}\xi=A\zeta-\hat{A}\eta. \tag{10}$$

Moreover, the values of the mean-square errors $\Delta(f,g;\hat{A}\xi)=\mathrm{E}|A\xi-\hat{A}\xi|^2$ and $\Delta(f,g;\hat{A}\eta)=\mathrm{E}|A\eta-\hat{A}\eta|^2$ of the estimates $\hat{A}\xi$ and $\hat{A}\eta$ respectively satisfy the equalities

$$\Delta\left(f,g;\hat{A}\xi\right)=\mathrm{E}\left|A\xi-\hat{A}\xi\right|^2=\mathrm{E}\left|A\zeta-A\eta-A\zeta+\hat{A}\eta\right|^2=\mathrm{E}\left|A\eta-\hat{A}\eta\right|^2=\Delta\left(f,g;\hat{A}\eta\right).$$

In this paper we do not try to find the optimal estimate $\hat{A}\eta$ by minimizing the mean-square error as a function of admissible estimates directly. We propose to use the properties of projections in the Hilbert space $H = L_2(\Omega, \mathrm{F}, \mathrm{P})$ of the random variables with zero mean values and finite second moments. The available observations $\{\xi^{(n)}(t,\tau)+\eta^{(n)}(t,\tau): t \leq 0\}$, $\tau > 0$, generate a closed linear subspace $H^0(\xi_\tau^{(n)}+\eta_\tau^{(n)})$ of the Hilbert space $H = L_2(\Omega, \mathrm{F}, \mathrm{P})$. Thus the optimal estimate $\hat{A}\eta$ is a projection of the element $A\eta$ of the space H on the subspace $H^0(\xi_\tau^{(n)}+\eta_\tau^{(n)})$ and the value of the mean-square error $\Delta(f,g;\hat{A}\eta)$ is the minimal distance from the element $A\eta$ from the space H to the subspace $H^0(\xi_\tau^{(n)}+\eta_\tau^{(n)})$. These properties give us two conditions characterizing the optimal estimate $\hat{A}\eta$:

1) $\hat{A}\eta \in H^0\left(\xi_\tau^{(n)}+\eta_\tau^{(n)}\right)$;
2) $\left(A\eta-\hat{A}\eta\right) \perp H^0\left(\xi_\tau^{(n)}+\eta_\tau^{(n)}\right)$.

The obtained conditions allow us to find the spectral characteristic $h_\tau(\lambda)$ of the estimate $\hat{A}\eta$. To be able to use these conditions we have to describe the spectral representations of the involved processes.

The stationary random process $\eta(t)$ with the spectral function $G(\lambda)$ and the nth increment $\eta^{(n)}(t,\tau)$ admits the spectral representations

$$\eta(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ_\eta(\lambda)$$

and

$$\eta^{(n)}(t,\tau) = \int_{-\infty}^{\infty} e^{i\lambda t}(1-e^{-i\lambda\tau})^n dZ_\eta(\lambda),$$

where $Z_\eta(\lambda)$ is a random process with independent increments defined on R corresponding to the spectral function $G(\lambda)$. The stationary increment process $\zeta^{(n)}(t,\tau)$ admits the spectral representation

$$\zeta^{(n)}(t,\tau) = \int_{-\infty}^{\infty} e^{i\lambda t}(1-e^{-i\lambda\tau})^n \frac{(1+i\lambda)^n}{(i\lambda)^n} dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda)$$

$$= \int_{-\infty}^{\infty} e^{i\lambda t}(1-e^{-i\lambda\tau})^n \frac{(1+i\lambda)^n}{(i\lambda)^n} dZ_{\xi^{(n)}}(\lambda) + \int_{-\infty}^{\infty} e^{i\lambda t}(1-e^{-i\lambda\tau})^n dZ_\eta(\lambda),$$

where $dZ_{\eta^{(n)}}(\lambda) = \dfrac{(i\lambda)^n}{(1+i\lambda)^n} dZ_\eta(\lambda)$, $\lambda \in \mathrm{R}$. From the described representations we can obtain the spectral density $p(\lambda)$ of the random process $\zeta(t)$:

$$p(\lambda) = f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda).$$

Finally, the spectral representation of the estimate $\hat{A}\eta$ is

$$\hat{A}\eta = \int_{-\infty}^{\infty} h_\tau(\lambda) dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda),$$

and the representation of the estimate $\hat{A}\xi$ of the functional $A\xi$ is

$$\hat{A}\xi = A\zeta - \int_{-\infty}^{\infty} h_\tau(\lambda) dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda). \tag{11}$$

It comes from the condition 2) that for all $t \le 0$ the function $h_\tau(\lambda)$ satisfies the following equality:

$$\int_{-\infty}^{\infty} \left[A(\lambda) \frac{(-i\lambda)^n}{(1-i\lambda)^n} g(\lambda) - h_\tau(\lambda) \left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda) \right) \right] \frac{(1-i\lambda)^n}{(-i\lambda)^n} (1-e^{i\lambda\tau})^n e^{-i\lambda t} d\lambda = 0, \tag{12}$$

where

$$A(\lambda) = \int_0^{\infty} a(t) e^{-i\lambda t} dt.$$

Let us define the function

$$C^\tau(\lambda) = \left[A(\lambda) \frac{(-i\lambda)^n}{(1-i\lambda)^n} g(\lambda) - h_\tau(\lambda) \left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda) \right) \right] \frac{(1-i\lambda)^n}{(-i\lambda)^n} (1-e^{i\lambda\tau})^n, \quad \lambda \in \mathbb{R},$$

and its Fourier transform

$$\mathbf{c}_\tau(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C^\tau(\lambda) e^{-i\lambda t} d\lambda, \quad t \in \mathbb{R}.$$

Condition (12) let us conclude that the function $\mathbf{c}_\tau(t)$ is equal to 0 on $(-\infty;0]$, which implies

$$C^\tau(\lambda) = \int_0^{\infty} \mathbf{c}_\tau(t) e^{i\lambda t} dt.$$

Hence the function $h_\tau(\lambda)$ is of the form

$$h_\tau(\lambda) = \frac{A(\lambda)(-i\lambda)^n g(\lambda)}{(1-i\lambda)^n \left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda) \right)} - \frac{(1-e^{i\lambda\tau})^{-n} (-i\lambda)^n C^\tau(e^{i\lambda})}{(1-i\lambda)^n \left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda) \right)}.$$

Denote by $L_2^0\left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda) \right)$ the closed linear subspace of the Hilbert space $L_2\left(f(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g(\lambda) \right)$ generated by the set of functions

$$\left\{e^{i\lambda t}(1-e^{-i\lambda\tau})^n\frac{(1+i\lambda)^n}{(i\lambda)^n}:t\leq 0\right\}.$$

It follows from the condition 1) and from isometry between the subspaces $L_2^0\left(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda)\right)$ and $H^0(\xi_\tau^{(n)}+\eta_\tau^{(n)})$ that the following properties of the spectral characteristic $h_\tau(\lambda)$ hold true:

$$h_\tau(\lambda)=h(\lambda)(1-e^{-i\lambda\tau})^n\frac{(1+i\lambda)^n}{(i\lambda)^n},\quad h(\lambda)=\int_{-\infty}^0 u(t)e^{i\lambda t}dt,$$

for some function $u(t)\in L_2^0$,

$$\int_{-\infty}^{\infty}|h_\tau(\lambda)|^2|1-e^{i\lambda\tau}|^{2n}\left(\frac{(1+\lambda^2)^n}{\lambda^{2n}}f(\lambda)+g(\lambda)\right)d\lambda<\infty,\quad \frac{(i\lambda)^n h_\tau(\lambda)}{(1+i\lambda)^n(1-e^{-i\lambda\tau})^n}\in L_2^0.$$

Thus for all $s\geq 0$ the following equality holds true

$$\int_{-\infty}^{\infty}\left[\frac{A(\lambda)(1-e^{-i\lambda\tau})^{-n}\lambda^{2n}g(\lambda)}{(1+\lambda^2)^n\left(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda)\right)}-\frac{|1-e^{i\lambda\tau}|^{-n}\lambda^{2n}C^\tau(\lambda)}{(1+\lambda)^{2n}\left(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda)\right)}\right]e^{-i\lambda s}d\lambda=0.$$

The last condition provides us the equation that determines the function $\mathbf{c}_\tau(t)$:

$$\int_{-\tau n}^{\infty}\mathbf{a}_\tau(t)\int_{-\infty}^{\infty}e^{-i\lambda(t+s)}\frac{\lambda^{2n}g(\lambda)}{|1-e^{i\lambda\tau}|^{2n}\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}d\lambda dt$$
$$=\int_0^{\infty}\mathbf{c}_\tau(t)\int_{-\infty}^{\infty}e^{i\lambda(t-s)}\frac{\lambda^{2n}}{|1-e^{i\lambda\tau}|^{2n}\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}d\lambda dt,\quad s\geq 0,\tag{13}$$

where

$$\mathbf{a}_\tau(t)=\sum_{l=\max\left\{0,\left[-\frac{t}{\tau}\right]'\right\}}^{n}\binom{n}{l}(-1)^l a(t+\tau l),\quad t\geq-\tau n.\tag{14}$$

Here by $[x]'$ we denote the least integer number among numbers which are greater than or equal to x. Equation (13) can be rewritten in terms of linear operators in the Hilbert space L_2. For the functions $\mathbf{x}(t)\in L_2[-\tau n;\infty)$ and $\mathbf{y}(t),\mathbf{z}(t)\in L_2[0;\infty)$ define the following linear operators:

$$(\mathbf{S}_\infty^\tau\mathbf{x})(s)=\frac{1}{2\pi}\int_{-\tau n}^{\infty}\mathbf{x}(t)\int_{-\infty}^{\infty}e^{-i\lambda(t+s)}\frac{\lambda^{2n}g(\lambda)}{|1-e^{i\lambda\tau}|^{2n}\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}d\lambda dt, s\in[0;\infty),$$

$$(\mathbf{P}_{\infty}^{\tau}\mathbf{y})(s)=\frac{1}{2\pi}\int_0^{\infty}\mathbf{y}(t)\int_{-\infty}^{\infty}e^{i\lambda(t-s)}\frac{\lambda^{2n}}{|1-e^{i\lambda\tau}|^{2n}\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}d\lambda dt, s\in[0;\infty),$$

$$(\mathbf{Q}_{\infty}\mathbf{z})(s)=\frac{1}{2\pi}\int_0^{\infty}\mathbf{z}(t)\int_{-\infty}^{\infty}e^{i\lambda(t-s)}\frac{(1+\lambda^2)^n f(\lambda)g(\lambda)}{(1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)}d\lambda dt, s\in[0;\infty).$$

With the help of these operators equation (13) can be written in the form

$$(\mathbf{S}_{\infty}^{\tau}\mathbf{a}_{\tau})(s)=(\mathbf{P}_{\infty}^{\tau}\mathbf{c}_{\tau})(s),\quad s\geq 0.$$

Let the operator $\mathbf{P}_{\infty}^{\tau}$ be invertible. Then the last equation has the following solution

$$\mathbf{c}_{\tau}(s)=((\mathbf{P}_{\infty}^{\tau})^{-1}\mathbf{S}_{\infty}^{\tau}\mathbf{a}_{\tau})(s),\quad s\geq 0.$$

The sufficient condition for the existing the inverse operator $(\mathbf{P}_{\infty}^{\tau})^{-1}$ and a formula for calculating this operator are discussed below in Lemma 3.1.

The obtained function $\mathbf{c}_{\tau}(s)$ allows us to write a solution to the filtering problem. The spectral characteristic $h_{\tau}(\lambda)$ of the optimal estimate $\hat{A}\xi$ of the functional $A\xi$ is calculated by the formula

$$h_{\tau}(\lambda)=\frac{A(\lambda)(1+i\lambda)^n(-i\lambda)^n g(\lambda)}{(1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)}-\frac{(1+i\lambda)^n(-i\lambda)^n C^{\tau}(e^{i\lambda})}{(1-e^{i\lambda\tau})^n\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)},\qquad (15)$$

where

$$C^{\tau}(\lambda)=\int_0^{\infty}((\mathbf{P}_{\infty}^{\tau})^{-1}\mathbf{S}_{\infty}^{\tau}\mathbf{a}_{\tau})(t)e^{i\lambda t}dt.$$

The value of the mean-square error of the estimate $\hat{A}\xi$ of the functional $A\xi$ is calculated by the formula

$$\Delta\left(f,g;\hat{A}\xi\right)=\Delta\left(f,g;\hat{A}\eta\right)=\mathrm{E}\left|A\eta-\hat{A}\eta\right|^2$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|A(\lambda)(1-e^{i\lambda\tau})^n(1+\lambda^2)^n f(\lambda)+\lambda^{2n}C^{\tau}(\lambda)\right|^2}{|1-e^{i\lambda\tau}|^{2n}(1+\lambda^2)^{2n}(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda))^2}g(\lambda)d\lambda$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|A(\lambda)(1-e^{i\lambda\tau})^n(-i\lambda)^n g(\lambda)-(-i\lambda)^n C^{\tau}(\lambda)\right|^2}{|1-e^{i\lambda\tau}|^{2n}(1+\lambda^2)^{n}(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda))^2}f(\lambda)d\lambda$$

$$=\langle\mathbf{S}_{\infty}^{\tau}\mathbf{a}_{\tau},(\mathbf{P}_{\infty}^{\tau})^{-1}\mathbf{S}_{\infty}^{\tau}\mathbf{a}_{\tau}\rangle+\langle\mathbf{Q}_{\infty}\mathbf{a},\mathbf{a}\rangle,\qquad (16)$$

where the function $\mathbf{a}(t)$, $t\geq 0$, is defined as $\mathbf{a}(t)=a(t)$.

Summing up our reasonings we can formulate the following theorem.

Theorem 3.1 *Consider a random process* $\{\xi(t), t \in \mathbb{R}\}$ *with stationary* n *th increments* $\xi^{(n)}(t,\tau)$ *which has the spectral density* $f(\lambda)$. *Let* $\{\eta(t), t \in \mathbb{R}\}$ *be a stationary process with the spectral density* $g(\lambda)$. *Assume that these processes are uncorrelated and their spectral densities satisfy the minimality conditions (8). If the function* $\mathbf{a}_\tau(t)$ *defined by (14) satisfies conditions (9), the optimal linear estimate* $\hat{A}\xi$ *of the functional* $A\xi$ *of the unknown values* $\xi(t)$, $t \le 0$, *from observations of the process* $\xi(t)+\eta(t)$ *at points* $t \le 0$ *is determined by formula (11). The spectral characteristic* $h_\tau(\lambda)$ *of the optimal estimate* $\hat{A}\xi$ *can be calculated by formula (15). The value of the mean-square error* $\Delta(f,g;\hat{A}\xi)$ *of the optimal estimate is calculated by formula (16).*

As a particular case of Theorem 3.1 we can derive the optimal estimate

$$\hat{A}_T\xi = A_T\zeta - \int_{-\infty}^{\infty} h_{\tau,T}(\lambda)dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda) \tag{17}$$

of the functional $A_T\xi = \int_0^T a(t)\xi(-t)dt$ from observations of the process $\xi(t)+\eta(t)$ at points $t \le 0$. Consider a function $a(t)$, $t \ge 0$, which is equal to 0 for $t > T$. In this case the formula for calculating the spectral characteristic $h_{\tau,T}(\lambda)$ of the linear estimate $\hat{A}_T\xi$ is the following:

$$h_{\tau,T}(\lambda) = \frac{A_T(\lambda)(1+i\lambda)^n(-i\lambda)^n g(\lambda)}{(1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)} - \frac{(1+i\lambda)^n(-i\lambda)^n C_T^\tau(e^{i\lambda})}{(1-e^{i\lambda\tau})^n\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}, \tag{18}$$

$$A_T(\lambda) = \int_0^T a(t)e^{-i\lambda t}dt, \quad C_T^\tau(\lambda) = \int_0^\infty ((\mathbf{P}_\infty^\tau)^{-1}\mathbf{S}_T^\tau\mathbf{a}_{\tau,T})(t)e^{i\lambda t}dt,$$

where the linear operator $\mathbf{S}_T^\tau$ is defined as

$$(\mathbf{S}_T^\tau\mathbf{x})(s) = \frac{1}{2\pi}\int_{-\tau n}^T \mathbf{x}(t)\int_{-\infty}^{\infty} e^{-i\lambda(t+s)} \frac{\lambda^{2n}g(\lambda)}{|1-e^{i\lambda\tau}|^{2n}\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}d\lambda dt, s \in [0;\infty),$$

and the function $\mathbf{a}_{\tau,T}(t)$, $t \in [-\tau n; T]$, is defined as

$$\mathbf{a}_{\tau,T}(t) = \sum_{l=\max\left\{0,\left[-\frac{t}{\tau}\right]\right\}}^{\min\left\{n,\left[\frac{T-t}{\tau}\right]\right\}} \binom{n}{l}(-1)^l a(t+\tau l), \quad t \in [-\tau n; T].$$

The mean-square error of the estimate $\hat{A}_T\xi$ is calculated by the formula

$$\Delta\left(f,g;\hat{A}_T\xi\right) = \Delta\left(f,g;\hat{A}_T\eta\right) = \mathrm{E}\left|A_T\eta - \hat{A}_T\eta\right|^2$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|A_T(\lambda)(1-e^{i\lambda\tau})^n(1+\lambda^2)^n f(\lambda)+\lambda^{2n}C_T^{\tau}(\lambda)\right|^2}{|1-e^{i\lambda\tau}|^{2n}\,(1+\lambda^2)^{2n}(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda))^2}g(\lambda)d\lambda$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|A_T(\lambda)(1-e^{i\lambda\tau})^n(-i\lambda)^n g(\lambda)-(-i\lambda)^n C_T^{\tau}(\lambda)\right|^2}{|1-e^{i\lambda\tau}|^{2n}\,(1+\lambda^2)^{n}(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda))^2}f(\lambda)d\lambda$$

$$=\langle \mathbf{S}_T^{\tau}\mathbf{a}_{\tau,T},(\mathbf{P}_{\infty}^{\tau})^{-1}\mathbf{S}_T^{\tau}\mathbf{a}_{\tau,T}\rangle+\langle \mathbf{Q}_T\mathbf{a}_T,\mathbf{a}_T\rangle, \qquad (19)$$

where the linear operator $\mathbf{Q}_T$ is defined as

$$(\mathbf{Q}_T\mathbf{z})(s)=\frac{1}{2\pi}\int_0^T\mathbf{z}(t)\int_{-\infty}^{\infty}e^{i\lambda(t-s)}\frac{(1+\lambda^2)^n f(\lambda)g(\lambda)}{(1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)}d\lambda dt, s\in[0;\infty),$$

and the function $\mathbf{a}_T(t)$, $t\in[0;T]$, is defined as $\mathbf{a}_T(t)=a(t)$.

We can conclude that the following theorem holds true.

Theorem 3.2 *Consider a random process* $\{\xi(t), t\in\mathbb{R}\}$ *with stationary* n *th increments* $\xi^{(n)}(t,\tau)$ *which has the spectral density* $f(\lambda)$. *Let* $\{\eta(t), t\in\mathbb{R}\}$ *be a stationary process with the spectral density* $g(\lambda)$. *Assume that these processes are uncorrelated and their spectral densities satisfy the minimality conditions (8). Then the optimal linear estimate* $\hat{A}_T\xi$ *of the functional* $A_T\xi$ *of unknown values* $\xi(t)$, $t\in[-T;0]$, *from observations of the process* $\xi(t)+\eta(t)$ *at points* $t\le 0$ *is determined by formula (17). The spectral characteristic* $h_{\tau,T}(\lambda)$ *of the optimal estimate* $\hat{A}_T\xi$ *can be calculated by formula (18). The value of the mean-square error* $\Delta(f,g;\hat{A}_T\xi)$ *of the optimal estimate is calculated by formula (19).*

In general case the problem of finding the inverse operator $(\mathbf{P}_{\infty}^{\tau})^{-1}$ is difficult. However, it can be solved if the increment process $\xi^{(n)}(t,\tau)+\eta^{(n)}(t,\tau)$ admits a one-sided moving average representation and the function $\frac{\lambda^{2n}}{|1-e^{i\lambda\tau}|^{2n}\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}$ has the following canonical factorization (see, for example, Gikhman and Skorohod, 2004):

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\tau}|^{2n}\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}=\left|\int_0^{\infty}\psi_{\tau}(t)e^{-i\lambda t}dt\right|^2=\left|\int_0^{\infty}\theta_{\tau}(t)e^{-i\lambda t}dt\right|^{-2}. \qquad (20)$$

Lemma 3.1 *Suppose that the function* $\frac{\lambda^{2n}}{|1-e^{i\lambda\tau}|^{2n}\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}$ *admits the canonical factorization (20). Consider linear operators* Ψ_{τ} *and* Θ_{τ} *in the space* $L_2([0;\infty))$ *with the kernels*

$\Psi_\tau(s,t)=\overline{\psi}_\tau(s-t)1_{s\geq t}$ *and* $\Theta_\tau(s,t)=\overline{\theta}_\tau(s-t)1_{s\geq t}$. *Then the linear operator* $\mathbf{P}_\infty^\tau$ *in the space* $L_2([0;\infty))$ *admits the factorization* $\mathbf{P}_\infty^\tau=\Psi_\tau^*\Psi_\tau$ *and the inverse operator* $(\mathbf{P}_\infty^\tau)^{-1}$ *admits the factorization* $(\mathbf{P}_\infty^\tau)^{-1}=\Theta_\tau\Theta_\tau^*$, *where the linear operators* Ψ_τ^* *and* Θ_τ^* *have the kernels* $\Psi_\tau(s,t)=\psi_\tau(t-s)1_{t\geq s}$ *and* $\Theta_\tau(s,t)=\theta_\tau(t-s)1_{t\geq s}$.

Proof. Factorization (20) implies the following relation:

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\tau}|^{2n}\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}=\int_{-\infty}^{\infty}p_\tau(t)e^{i\lambda t}dt=\left|\int_0^\infty\psi_\tau(t)e^{-i\lambda t}dt\right|^2$$
$$=\int_{-\infty}^{0}\int_{-z}^{\infty}\psi_\tau(t)\overline{\psi}_\tau(t+z)e^{i\lambda z}dtdz+\int_0^\infty\int_0^\infty\psi_\tau(t)\overline{\psi}_\tau(t+z)e^{i\lambda z}dtdz.$$

Thus,

$$p_\tau(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\lambda z}\lambda^{2n}}{|1-e^{i\lambda\tau}|^{2n}\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}d\lambda=\int_0^\infty\psi_\tau(t)\overline{\psi}_\tau(t+z)dt,\quad t\geq 0,$$

and $p(-z)=\overline{p_\tau(z)}$, $z\geq 0$. In the case of $s\geq t$ the kernel of the operator $\mathbf{P}_\infty^\tau$ is calculated by the formula

$$P_\infty^\tau(s,t)=\int_s^\infty\psi_\tau(z-s)\overline{\psi}_\tau(z-t)dz=(\Psi_\tau^*\Psi_\tau)(s,t),$$

and in the case of $s<t$ the kernel of the operator $\mathbf{P}_\infty^\tau$ is calculated by the formula

$$P_\infty^\tau(s,t)=\overline{P_\infty^\tau(t,s)}=\int_t^\infty\overline{\psi}_\tau(z-s)\psi_\tau(z-t)dz=(\Psi_\tau^*\Psi_\tau)(s,t).$$

These formulas imply the factorization $\mathbf{P}_\infty^\tau=\Psi_\tau^*\Psi_\tau$ and the relation

$$P_\infty^\tau(s,t)=\int_{\max\{t,s\}}^\infty\psi_\tau(z-s)\overline{\psi}_\tau(z-t)dz,\quad s,t\geq 0.$$

The factorization $(\mathbf{P}_\infty^\tau)^{-1}=\Theta_\tau\Theta_\tau^*$ comes from the equality $\Psi_\tau\Theta_\tau=\Theta_\tau\Psi_\tau=I$ which has to be proved. Factorization (20) implies $|\Xi_\tau(\lambda)|^2=1$, where

$$\Xi_\tau(\lambda)=\left(\int_0^\infty\psi_\tau(t)e^{-i\lambda t}dt\right)\left(\int_0^\infty\theta_\tau(t)e^{-i\lambda t}dt\right)=\int_0^\infty\left(\int_0^z\psi_\tau(t)\theta_\tau(z-t)dt\right)e^{-i\lambda z}dz.\qquad(21)$$

Therefore, the following relations hold true:

$$(\overline{\Theta}_\tau\overline{\Psi}_\tau\mathbf{x})(z)=\int_0^z\theta_\tau(z-t)\int\left(\int_0^t\psi_\tau(t-s)\mathbf{x}(s)ds\right)dt=\int_0^z\mathbf{x}(s)\left(\int_s^z\psi_\tau(t-s)\theta_\tau(z-t)dt\right)ds$$
$$=\int_0^z\mathbf{x}(s)\left(\int_0^{z-s}\psi_\tau(q)\theta_\tau(z-s-q)dq\right)ds=\mathbf{x}(z).$$

Lemma is proved.

4. Minimax-Robust Method of Filtering

In the previous section we solved the filtering problem for the functionals $A\xi$ and $A_T\xi$ which depend on the unknown values of the random process $\xi(t)$ based on observations of the random process $\xi(t)+\eta(t)$. The derived formulas (15) and (18) for the spectral characteristics of the mean-square estimates $\hat{A}\xi$ and $\hat{A}_T\xi$ can be used only under the assumption that we know the spectral densities $f(\lambda)$ and $g(\lambda)$ of the random processes $\xi(t)$ and $\eta(t)$. In practice,however, we do not have the exact values of these spectral densities. For this reason the minimax (robust) approach to estimation of functionals $A\xi$ and $A_T\xi$ is suitable. This method allows us to find estimates that minimize the maximum values of the mean-square errors of the estimates for all spectral densities from a given class $\mathsf{D}=\mathsf{D}_f\times\mathsf{D}_g$ of admissible spectral densities.

Definition 4.1 *For a given class of spectral densities* $\mathsf{D}=\mathsf{D}_f\times\mathsf{D}_g$ *the spectral densities* $f^0(\lambda)\in\mathsf{D}_f$ *and* $g^0(\lambda)\in\mathsf{D}_g$ *are called least favorable in the class* D *for the optimal linear filtering of the functional* $A\xi$ *if the following relation holds true*

$$\Delta(f^0,g^0)=\Delta\left(h(f^0,g^0);f^0,g^0\right)=\max_{(f,g)\in\mathsf{D}_f\times\mathsf{D}_g}\Delta\left(h(f,g);f,g\right).$$

Definition 4.2 *For a given class of spectral densities* $\mathsf{D}=\mathsf{D}_f\times\mathsf{D}_g$ *the spectral characteristic* $h^0(\lambda)$ *of the optimal linear estimate of the functional* $A\xi$ *is called minimax-robust if the following conditions are satisfied:*

$$h^0(\lambda)\in H_{\mathsf{D}}=\bigcap_{(f,g)\in\mathsf{D}_f\times\mathsf{D}_g}L_2^0\left(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda)\right),$$

$$\min_{h\in H_{\mathsf{D}}}\max_{(f,g)\in\mathsf{D}_f\times\mathsf{D}_g}\Delta(h;f,g)=\max_{(f,g)\in\mathsf{D}_f\times\mathsf{D}_g}\Delta(h^0;f,g).$$

From the proposed in the previous section formulas and the introduced definitions we can obtain the following statement.

Lemma 4.1 *The spectral densities* $f^0\in\mathsf{D}_f$, $g^0\in\mathsf{D}_g$ *satisfying the minimality conditions (8) are least favorable in the class* $\mathsf{D}=\mathsf{D}_f\times\mathsf{D}_g$ *for the optimal linear filtering of the functional* $A\xi$ *if operators* $(\mathbf{P}_\infty^\tau)^0$, $(\mathbf{S}_\infty^\tau)^0$, $(\mathbf{Q}_\infty)^0$ *determined by the Fourier transformations of the functions*

$$\frac{\lambda^{2n}|1-e^{i\lambda\tau}|^{-2n}}{(1+\lambda^2)^n f^0(\lambda)+\lambda^{2n}g^0(\lambda)},\quad \frac{\lambda^{2n}g^0(\lambda)|1-e^{i\lambda\tau}|^{-2n}}{(1+\lambda^2)^n f^0(\lambda)+\lambda^{2n}g^0(\lambda)},\quad \frac{(1+\lambda^2)^n f^0(\lambda)g^0(\lambda)}{(1+\lambda^2)^n f^0(\lambda)+\lambda^{2n}g^0(\lambda)}$$

give a solution of the conditional extremum problem

$$\max_{(f,g)\in D_f\times D_g}\left(\langle \mathbf{S}^\tau_\infty \mathbf{a}_\tau,(\mathbf{P}^\tau_\infty)^{-1}\mathbf{S}^\tau_\infty \mathbf{a}_\tau\rangle+\langle \mathbf{Q}_\infty \mathbf{a},\mathbf{a}\rangle\right)=\langle (\mathbf{S}^\tau_\infty)^0 \mathbf{a}_\tau,((\mathbf{P}^\tau_\infty)^0)^{-1}(\mathbf{S}^\tau_\infty)^0 \mathbf{a}_\tau\rangle+\langle (\mathbf{Q}_\infty)^0 \mathbf{a},\mathbf{a}\rangle. \tag{22}$$

The minimax spectral characteristic is determined as $h^0=h_\tau(f^0,g^0)$ *if* $h_\tau(f^0,g^0)\in H_D$.

The minimax-robust spectral characteristic h^0 and the least favorable spectral densities (f^0,g^0) form a saddle point of the function $\Delta(h;f,g)$ on the set $H_D\times D$. The saddle point inequalities

$$\Delta(h;f^0,g^0)\ge\Delta(h^0;f^0,g^0)\ge\Delta(h^0;f,g)\quad \forall f\in D_f,\forall g\in D_g,\forall h\in H_D$$

hold true if $h^0=h_\tau(f^0,g^0)$ and $h_\tau(f^0,g^0)\in H_D$, where the pair (f^0,g^0) provides a solution to the following constraint optimization problem:

$$\hat{\Delta}(f,g)=-\Delta\left(h_\tau(f^0,g^0);f,g\right)\to\inf,\quad (f,g)\in D_f\times D_g,$$

$$\Delta\left(h_\tau(f^0,g^0);f,g\right)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|A(\lambda)(1-e^{i\lambda\tau})^n(1+\lambda^2)^n f^0(\lambda)+\lambda^{2n}\int_0^\infty(((\mathbf{P}^\tau_\infty)^0)^{-1}(\mathbf{S}^\tau_\infty)^0\mathbf{a}_\tau)(t)e^{i\lambda t}dt\right|^2}{|1-e^{i\lambda\tau}|^{2n}(1+\lambda^2)^{2n}(f^0(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g^0(\lambda))^2}g(\lambda)d\lambda$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|A(\lambda)(1-e^{i\lambda\tau})^n(-i\lambda)^n g^0(\lambda)-(-i\lambda)^n\int_0^\infty(((\mathbf{P}^\tau_\infty)^0)^{-1}(\mathbf{S}^\tau_\infty)^0\mathbf{a}_\tau)(t)e^{i\lambda t}dt\right|^2}{|1-e^{i\lambda\tau}|^{2n}(1+\lambda^2)^{n}(f^0(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g^0(\lambda))^2}f(\lambda)d\lambda.$$

This optimization problem is equivalent to the unconditional extremum problem

$$\Delta_D(f,g)=\hat{\Delta}(f,g)+\delta(f,g\,|\,D_f\times D_g)\to\inf,$$

where $\delta(f,g|D_f\times D_g)$ is the indicator function of the set $D_f\times D_g$. Solution (f^0,g^0) to this unconditional extremum problem is characterized by the condition $0\in\partial\Delta_D(f^0,g^0)$, where $\partial\Delta_D(f^0,g^0)$ is the subdifferential of the functional $\Delta_D(f,g)$ at point $(f^0,g^0)\in D=D_f\times D_g$, that is the set of all continuous linear functionals Λ on $L_1\times L_1$ which satisfy an inequality

$$\Delta_D(f,g)-\Delta_D(f^0,g^0)\ge\Lambda((f,g)-(f^0,g^0)),\quad (f,g)\in D.$$

For more details we refer to the books by Rockafellar (1997), Ioffe and Tihomirov (1979) Moklyachuk (2008b) and Pshenychnyi (1982).

Note, that the form of the functional $\Delta\left(h_\tau(f^0,g^0);f,g\right)$ is convenient for application the Lagrange method of indefinite multipliers for finding solution to the extremum problem (23). Making use the method of Lagrange multipliers and the form of subdifferentials of the indicator functions we describe relations that determine the least favourable spectral densities in some special classes of spectral densities (for additional details see books by Moklyachuk, 2008a; Moklyachuk and Masyutka, 2012; Golichenko and Moklyachuk, 2014).

5. Least Favorable Spectral Densities in the Class $D_f^0 \times D_g^0$

Consider the optimal linear filtering problem for the functional $A\xi$ in the case where the set of admissible spectral densities is of the form $D = D_f^0 \times D_g^0$, where

$$D_f^0 = \left\{ f(\lambda) \,|\, \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)d\lambda \le P_1 \right\}, \quad D_g^0 = \left\{ g(\lambda) \,|\, \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\lambda)d\lambda \le P_2 \right\}.$$

Assume that spectral densities $f^0 \in D_f^0$, $g^0 \in D_g^0$ and the functions

$$h_{\tau,f}(f^0,g^0) = \frac{\left| A(\lambda)(1-e^{i\lambda\tau})^n(-i\lambda)^n g^0(\lambda) - (-i\lambda)^n \int_0^{\infty} (((\mathbf{P}_\infty^\tau)^0)^{-1}(\mathbf{S}_\infty^\tau)^0 \mathbf{a}_\tau)(t) e^{i\lambda t} dt \right|}{|1-e^{i\lambda\tau}|^n |1+i\lambda|^n \left(f^0(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g^0(\lambda) \right)}, \tag{23}$$

$$h_{\tau,g}(f^0,g^0) = \frac{\left| A(\lambda)(1-e^{i\lambda\tau})^n(1+\lambda^2)^n f^0(\lambda) + \lambda^{2n} \int_0^{\infty} (((\mathbf{P}_\infty^\tau)^0)^{-1}(\mathbf{S}_\infty^\tau)^0 \mathbf{a}_\tau)(t) e^{i\lambda t} dt \right|}{|1-e^{i\lambda\tau}|^n (1+\lambda^2)^n \left(f^0(\lambda) + \frac{\lambda^{2n}}{(1+\lambda^2)^n} g^0(\lambda) \right)} \tag{24}$$

are bounded. Under these conditions the functional $\Delta(h_\tau(f^0,g^0); f,g)$ is continuous and bounded in the $L_1 \times L_1$ space and we can apply the Lagrange multipliers method to derive that the least favorable densities $f^0 \in D_f^0$, $g^0 \in D_g^0$ satisfy the equations

$$\left| A(\lambda)(1-e^{i\lambda\tau})^n(1+\lambda^2)^n f^0(\lambda) + \lambda^{2n} \int_0^{\infty} (((\mathbf{P}_\infty^\tau)^0)^{-1}(\mathbf{S}_\infty^\tau)^0 \mathbf{a}_\tau)(t) e^{i\lambda t} dt \right| = \alpha_1 |1-e^{i\lambda\tau}|^n \left((1+\lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda) \right), \tag{25}$$

$$\left| A(\lambda)(1-e^{i\lambda\tau})^n(-i\lambda)^n g^0(\lambda) - (-i\lambda)^n \int_0^{\infty} (((\mathbf{P}_\infty^\tau)^0)^{-1}(\mathbf{S}_\infty^\tau)^0 \mathbf{a}_\tau)(t) e^{i\lambda t} dt \right| = \alpha_2 |1-e^{i\lambda\tau}|^n |1-i\lambda|^{-n} \left((1+\lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda) \right), \tag{26}$$

where $\alpha_1 \ge 0$ and $\alpha_2 \ge 0$ are constants such that $\alpha_1 \ne 0$ if $(2\pi)^{-1}\int_{-\infty}^{\infty} f^0(\lambda)d\lambda = P_1$ and $\alpha_2 \ne 0$ if $(2\pi)^{-1}\int_{-\infty}^{\infty} g^0(\lambda)d\lambda = P_2$.

Thus, the following statements hold true.

Theorem 5.1 *Let conditions (9) be satisfied, let the spectral densities* $f^0(\lambda)\in D_f^0$, $g^0(\lambda)\in D_g^0$ *satisfy conditions (8) and let the functions* $h_{\tau,f}(f^0,g^0)$, $h_{\tau,g}(f^0,g^0)$ *determined by formulas (23), (24) be bounded. The spectral densities* $f^0(\lambda)$ *and* $g^0(\lambda)$ *are the least favorable in the class* $D=D_f^0\times D_g^0$ *for the optimal linear estimation of the functional* $A\xi$ *if they satisfy equations (25), (26) and determine a solution to the extremum problem (22). The minimax-robust spectral characteristic* $h_\tau(f^0,g^0)$ *of the optimal estimate of the functional* $A\xi$ *is determined by formula (15).*

Theorem 5.2 *Let the spectral density* $f(\lambda)$ *be known. Let the density* $f(\lambda)$ *and the spectral density* $g^0(\lambda)\in D_g^0$ *satisfy conditions (8). Let conditions (9) be satisfied, and let the function* $h_{\tau,g}(f,g^0)$ *determined by formula (24) be bounded. Then the spectral density* $g^0(\lambda)$ *is the least favorable in the class* D_g^0 *for the optimal linear filtering of the functional* $A\xi$ *if it has the form*

$$g^0(\lambda)=(1+\lambda^2)^n\lambda^{-2n}\max\{0,\,f_1(\lambda)\},$$
$$f_1(\lambda)=\frac{\left|A(\lambda)(1-e^{i\lambda\tau})^n(1+\lambda^2)^n f(\lambda)+\lambda^{2n}\int_0^\infty(((\mathbf{P}_\infty^\tau)^0)^{-1}(\mathbf{S}_\infty^\tau)^0\mathbf{a}_\tau)(t)e^{i\lambda t}dt\right|}{\alpha_2\,|1-e^{i\lambda\tau}|^n\,(1+\lambda^2)^n}-f(\lambda) \tag{27}$$

and the pair (f,g^0) *determines a solution to the extremum problem (22). The minimax-robust spectral characteristic* $h_\tau(f,g^0)$ *of the optimal estimate of the functional* $A\xi$ *is determined by formula (15).*

Theorem 5.3 *Let the spectral density* $g(\lambda)$ *be known. Let the spectral density* $f^0(\lambda)\in D_f^0$ *and a known spectral density* $g(\lambda)$ *satisfy conditions (8). Let conditions (9) be satisfied, and let the function* $h_{\tau,f}(f^0,g)$ *determined by formula (23) be bounded. Then the spectral density* $f^0(\lambda)$ *is the least favorable in the class* D_f^0 *for the optimal linear filtering of the functional* $A\xi$ *if it has the form*

$$f^0(\lambda)=\max\{0,g_1(\lambda)\},$$
$$g_1(\lambda)=\frac{\left|A(\lambda)(1-e^{i\lambda\tau})^n(-i\lambda)^n g(\lambda)-(-i\lambda)^n\int_0^\infty(((\mathbf{P}_\infty^\tau)^0)^{-1}(\mathbf{S}_\infty^\tau)^0\mathbf{a}_\tau)(t)e^{i\lambda t}dt\right|}{\alpha_1\,|1-e^{i\lambda\tau}|^n\,(1+\lambda^2)^{n/2}}-\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda) \tag{28}$$

and the pair (f^0, g) determines a solution to the extremum problem (22). The minimax-robust spectral characteristic $h_\tau(f^0, g)$ of the optimal estimate of the functional $A\xi$ is determined by formula (15).

6. Least Favorable Densities in the Class $\mathsf{D} = \mathsf{D}_u^v \times \mathsf{D}_\varepsilon$

Consider the optimal linear filtering problem for the functional $A\xi$ in the case where the set of admissible spectral densities is of the form $\mathsf{D} = \mathsf{D}_u^v \times \mathsf{D}_\varepsilon$, where

$$\mathsf{D}_u^v = \left\{ f(\lambda) \,|\, v(\lambda) \le f(\lambda) \le u(\lambda), \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)d\lambda \le P_1 \right\},$$

$$\mathsf{D}_\varepsilon = \left\{ g(\lambda) \,|\, g(\lambda) = (1-\varepsilon)g_2(\lambda) + \varepsilon w(\lambda), \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\lambda)d\lambda \le P_2 \right\}.$$

We suppose that the spectral densities $u(\lambda)$, $v(\lambda)$ and $g_2(\lambda)$ from the definition of the set $\mathsf{D} = \mathsf{D}_u^v \times \mathsf{D}_\varepsilon$ are known, fixed and, in addition, the spectral densities $u(\lambda)$ and $v(\lambda)$ are bounded.

Let functions $h_{\tau,f}(f^0, g^0)$ and $h_{\tau,g}(f^0, g^0)$ determined by formulas (23), (24) be bounded for spectral densities $f^0 \in \mathsf{D}_u^v$, $g^0 \in \mathsf{D}_\varepsilon$. Condition $0 \in \partial\Delta_{\mathsf{D}}(f^0, g^0)$ implies the following equations that determine the least favorable spectral densities:

$$\left| A(\lambda)(1-e^{i\lambda\tau})^n (1+\lambda^2)^n f^0(\lambda) + \lambda^{2n}\int_0^\infty (((\mathbf{P}_\infty^\tau)^0)^{-1}(\mathbf{S}_\infty^\tau)^0 \mathbf{a}_\tau)(t)e^{i\lambda t}dt \right|$$
$$= |1-e^{i\lambda\tau}|^n \left((1+\lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda)\right)(\gamma_1(\lambda) + \gamma_2(\lambda) + \alpha_1), \qquad (29)$$

$$\left| A(\lambda)(1-e^{i\lambda\tau})^n (-i\lambda)^n g^0(\lambda) - (-i\lambda)^n \int_0^\infty (((\mathbf{P}_\infty^\tau)^0)^{-1}(\mathbf{S}_\infty^\tau)^0 \mathbf{a}_\tau)(t)e^{i\lambda t}dt \right|$$
$$= |1-e^{i\lambda\tau}|^n |1-i\lambda|^{-n} \left((1+\lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda)\right)(\varphi(\lambda) + \alpha_2), \qquad (30)$$

where $\gamma_1(\lambda) \le 0$ and $\gamma_1(\lambda) = 0$ if $f^0(\lambda) \ge v(\lambda)$; $\gamma_2(\lambda) \ge 0$ and $\gamma_2(\lambda) = 0$ if $f^0(\lambda) \le u(\lambda)$; $\varphi(\lambda) \le 0$ and $\varphi(\lambda) = 0$ when $g^0(\lambda) \ge (1-\varepsilon)g_1(\lambda)$.

The following statements hold true.

Theorem 6.1 *Let conditions (9) be satisfied. Let the spectral densities $f^0(\lambda) \in \mathsf{D}_u^v$, $g^0(\lambda) \in \mathsf{D}_\varepsilon$ satisfy conditions (8) and let the functions $h_{\tau,f}(f^0, g^0)$, $h_{\tau,g}(f^0, g^0)$ determined by formulas (23), (24) be bounded. Spectral densities $f^0(\lambda)$ and $g^0(\lambda)$ are the least favorable in the class $\mathsf{D} = \mathsf{D}_u^v \times \mathsf{D}_\varepsilon$ for the optimal linear filtering of the functional $A\xi$ if they satisfy equations (29), (30) and determine a solution to the extremum problem (22). The minimax-robust spectral characteristic $h_\tau(f^0, g^0)$ of the optimal estimate of the functional $A\xi$ is determined by formula (15).*

Theorem 6.2 *Let the spectral density* $f(\lambda)$ *be known. Let the density* $f(\lambda)$ *and the spectral density* $g^0(\lambda) \in D_g^0$ *satisfy conditions (8). Let conditions (9) be satisfied, and let the function* $h_{\tau,g}(f, g^0)$ *determined by formula (24) be bounded. Spectral density* $g^0(\lambda)$ *is the least favorable in the class* D_ε *for the optimal linear filtering of the functional* $A\xi$ *if it has the form*

$$g^0(\lambda) = \max\left\{(1-\varepsilon)g_2(\lambda), (1+\lambda^2)^n \lambda^{-2n} f_1(\lambda)\right\},$$

where the function $f_1(\lambda)$ *is defined by formula (27), and the pair* (f, g^0) *determines a solution to the extremum problem (22). The minimax-robust spectral characteristic* $h_\tau(f, g^0)$ *of the optimal estimate of the functional* $A\xi$ *is determined by formula (15).*

Theorem 6.3 *Let the spectral density* $g(\lambda)$ *be known. Let the spectral density* $f^0(\lambda) \in D_f^0$ *and the known spectral density* $g(\lambda)$ *satisfy conditions (8). Let conditions (9) be satisfied, and the function* $h_{\tau,f}(f^0, g)$ *determined by formula (23) be bounded. Spectral density* $f^0(\lambda)$ *is the least favorable in the class* D_u^v *for the optimal linear filtering of the functional* $A\xi$ *if it has the form*

$$f^0(\lambda) = \min\left\{v(\lambda), \max\left\{u(\lambda), g_1(\lambda)\right\}\right\},$$

where function $g_1(\lambda)$ *is defined by formula (28), and the pair* (f^0, g) *determines a solution to the extremum problem (22). The minimax-robust spectral characteristic* $h_\tau(f^0, g)$ *of the optimal estimate of the functional* $A\xi$ *is determined by formula (15).*

7. Conclusions

In this paper we apply two methods to find solution of the problem of mean-square optimal estimation of the linear functionals $A\xi = \int_0^\infty a(t)\xi(-t)dt$ and $A_T\xi = \int_0^T a(t)\xi(-t)dt$ which depend on the unknown values of a random process $\xi(t)$ with n th stationary increments from observations of the process $\xi(t) + \eta(t)$ at points $t \le 0$. The first method provides us with formulas for calculating the values of the mean-square errors and the spectral characteristics of the estimates of the functionals $A\xi$ and $A_T\xi$ in the case where the spectral densities of the processes are given. The second one, called minimax-robust method, let us solve the problem in the case where the spectral densities are not known, but a set of admissible spectral densities is available.

References

Dubovets'ka, I. I., & Moklyachu, M.P. (2013b). Extrapolation of periodically correlated processes from observations with noise. Theor. Probability and Math. Statist, 88, 43-55.

Dubovets'ka, I. I., & Moklyachuk, M.P. (2013a). Filtration of linear functionals of periodically correlated sequences. Theor. Probability and Math. Statist, 86, 51-64. http://dx.doi.org/10.1090/S0094-9000-2013-00888-6

Dubovets'ka, I. I., & Moklyachuk, M.P. (2013c). Minimax estimation problem for periodically correlated stochastic processes. Journal of Mathematics and System Science, 3(1), 26-30.

Dubovets'ka, I. I., & Moklyachuk, M.P. (2014). On minimax estimation problems for periodically correlated stochastic processes., Contemporary Mathematics and Statistics, 2(1), 123-150. http://dx.doi.org/10.7726/cms.2014.1001

Dubovets'ka, I.I., Masyutka, O.Yu., & Moklyachuk, M.P. (2012). Interpolation of periodically correlated stochastic sequences. Theor. Probability and Math. Statist, 84, 43-56. http://dx.doi.org/10.1090/S0094-9000-2012-00862-4

Franke, J. (1985). Minimax robust prediction of discrete time series. Z. Wahrsch. Verw. Gebiete, 68, 337-364. http://dx.doi.org/10.1007/BF00532645

Franke, J., & Poor, H. V. (1984). Minimax-robust filtering and finite-length robust predictors. Robust and Nonlinear Time Series Analysis. Lecture Notes in Statistics, Springer-Verlag, 26, 87-126. http://dx.doi.org/10.1007/978-1-4615-7821-5_6

Gikhman, I. I., & Skorokhod, A. V. (2004). The theory of stochastic processes. I. Springer, Berlin.

Grenander, U. (1957). A prediction problem in game theory. Ark. Mat,. 3, 371-379. http://dx.doi.org/10.1007/BF02589429

Ioffe, A. D., & Tihomirov, V. M. (1979). Theory of extremal problems. Amsterdam, New York, Oxford: North—Holland Publishing Company.

Karhunen, K. (1947). Uber lineare Methoden in der Wahrscheinlichkeitsrechnung. Annales Academiae Scientiarum Fennicae. Series A I. Mathematica, 37, 3-79.

Kassam, S.A., & Poor, H.V. (1985). Robust techniques for signal processing: A survey. Proceedings of the IEEE, 73, 433-481. http://dx.doi.org/10.1109/PROC.1985.13167

Kolmogorov, A.N. (1992). Selected works of A. N. Kolmogorov. Vol. II: Probability theory and mathematical statistics. Ed. by A. N. Shiryayev. Mathematics and Its Applications. Soviet Series. 26. Dordrecht etc.: Kluwer Academic Publishers, Moskva.

Luz, M. M., & Moklyachuk, M. P. (2012). Interpolation of functionals of stochastic sequences with stationary increments for observations with noise. Prykl. Stat., Aktuarna Finans. Mat., (2), 131- 148.

Luz, M. M., & Moklyachuk, M. P. (2013b). Minimax-robust filtering problem for stochastic sequence with stationary increments. Theor. Probability and Math. Stat., 89, 117 - 131.

Luz, M. M., Moklyachuk, M. P. (2013a). Interpolation of functionals of stochastic sequences with stationary increments. Theor. Probability and Math. Stat., 87, 117-133. http://dx.doi.org/10.1090/S0094-9000-2014-00908-4

Luz, M., & Moklyachuk, M. (2014a). Robust extrapolation problem for stochastic processes with stationary increments. Mathematics and Statistics, 2(2), 78-88.

Luz, M., & Moklyachuk, M. (2014b). Minimax-robust filtering problem for stochastic sequences with stationary increments and cointegrated sequences. Statistics, Optimization & Information Computing, 2(3), 176-199.

Luz, M., & Moklyachuk, M. (2015a). Minimax interpolation problem for random processes with stationary increments. Statistics, Optimization & Information Computing, 3(1), 30-41.

Luz, M., & Moklyachuk, M. (2015b). Minimax-robust prediction problem for stochastic sequences with stationary increments and cointegrated sequences. Statistics, Optimization & Information Computing, 3(2), 160-188.

Moklyachuk, M. P. (1990). Minimax extrapolation and autoregressive-moving average processes. Theory Probab. Math. Stat., 41, 77-84.

Moklyachuk, M. P. (2000). Robust procedures in time series analysis. Theory Stoch. Process., 6(3-4), 127-147.

Moklyachuk, M. P. (2001). Game theory and convex optimization methods in robust estimation problems. Theory Stoch. Process., 7(1-2), 253-264.

Moklyachuk, M. P. (2008a). Robust estimations of functionals of stochastic processes. Kyivskyi University, Kyiv.

Moklyachuk, M. P. (2008b). Nonsmooth analysis and optimization. Kyiv University, Kyiv.

Moklyachuk, M., & Luz, M. (2013). Robust extrapolation problem for stochastic sequences with stationary increments. Contemporary Mathematics and Statistics, 1(3), 123-150. http://dx.doi.org/10.7726/cms.2013.1009

Moklyachuk, M., & Masyutka, O. (2006b). Robust estimation problems for stochastic processes. Theory Stoch. Process., 12(3-4), 88-113.

Moklyachuk, M., & Masyutka, O. (2007). Robust filtering of stochastic processes. Theory Stoch. Process., 13(1-2), 166-181.

Moklyachuk, M., & Masyutka, O. (2008). Minimax prediction problem for multidimensional stationary stochastic sequences. Theory Stoch. Process., 14(3-4), 89-103.

Moklyachuk, M., & Masyutka, O. (2011). Minimax prediction problem for multidimensional stationary stochastic processes. Communications in Statistics – Theory and Methods, 40(19-20), 3700-3710. http://dx.doi.org/10.1080/03610926.2011.581190

Moklyachuk, M., & Masyutka, O. (2012). Minimax-robust estimation technique for stationary stochastic processes. LAP LAMBERT Academic Publishing.

Moklyachuk, M.P., & Masyutka, O.Yu. (2006a). Extrapolation of multidimensional stationary processes. Random Operators Stochastic Equations, 14(3), 233-244. http://dx.doi.org/10.1515/156939706778239819

Pinsker, M. S. (1955). The theory of curves with nth stationary increments in Hilbert spaces. Izvestiya Akademii Nauk SSSR. Ser. Mat., 19(5), 319-344.

Pinsker, M. S., & Yaglom, A. M. (1954). On linear extrapolation of random processes with nth stationary increments. Doklady Akademii Nauk SSSR, 94, 385-388.

Pshenichnyi, B.N. (1982). Necessary conditions for an extremum. 2nd ed. Nauka, Moskva.

Rockafellar, R. T. (1997). Convex Analysis. Princeton University Press.

Rozanov, Yu. A. (1990). Stationary stochastic processes. 2nd rev. ed. Nauka, Moskva.

Vastola, K. S., & Poor, H. V. (1983). An analysis of the effects of spectral uncertainty on Wiener filtering. Automatica, 28, 289-293. http://dx.doi.org/10.1016/0005-1098(83)90105-X

Wiener, N. (1966). Extrapolation, interpolation and smoothing of stationary time series. With engineering applications. The M. I. T. Press, Massachusetts Institute of Technology, Cambridge, Mass.

Yaglom, A. M. (1955). Correlation theory of stationary and related random processes with stationary nth increments. Mat. Sbornik, 37(1), 141-196.

Yaglom, A. M. (1957). Some classes of random fields in n-dimensional space related with random stationary processes. Teor. Veroyatn. Primen., 2, 292-338. http://dx.doi.org/10.1137/1102021

Yaglom, A. M. (1987a). Correlation theory of stationary and related random functions. Vol. 1: Basic results. Springer Series in Statistics, Springer-Verlag, New York etc.

Yaglom, A. M. (1987b). Correlation theory of stationary and related random functions. Vol. 2: Supplementary notes and references. Springer Series in Statistics, Springer-Verlag, New York etc.