On Supremum Distribution of Averaged Deviations of Random Orlicz Processes from the Class $\Delta^2$

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Abstract

This paper is devoted to investigation of supremum of averaged deviations over some continuous function of a stochastic process from Orlicz space of random variables specified by an Orlicz function from the class $\Delta^2$. An estimate of distribution of supremum of deviations $|X(t) - f(t)|$ is derived using method of majorizing measures. A special case of sub-Gaussian space of random variables is considered.

Keywords: Orlicz space; Orlicz process; Supremum distribution; Method of majorizing measure

1. Introduction

This paper is devoted to investigation of supremum of averaged deviations of stochastic process from Orlicz spaces of random variables over a continuous increasing function using method of majorizing measures. Namely, the following functional is studied

$$\sup_{t \in T} \left| X(t) - f(t) - \frac{1}{\mu(T)} \int_T X(u) - f(u) \, d\mu(u) \right|$$

where $(T, B, \mu)$ is a measurable space with a finite measure $\mu(T) < \infty$ and $f(u)$ is a given function. Using obtained with probability one estimates for the above functional we obtain an upper bound estimate for the distribution of supremum $\sup_{t \in T} |X(t) - f(t)|$.

Similar problems considered for different classes of Orlicz processes were investigated by many authors (Kozachenko and Ryazantseva, 1992; Kozachenko and Moklyachuk, 2003; Kozachenko et al., 2005; Kozachenko and Sergienko, 2014; Kozachenko and Mlavets, 2015; Yamnenko, 2015, 2016).

In particular, Kozachenko and Ryazantseva (1992) obtained conditions of boundedness and sample

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path continuity with probability one of stochastic processes from the Orlicz space of random variables generated by exponential Orlicz functions. Kozachenko and Sergiienko (2014) constructed tests for a hypothesis concerning the form of the covariance function of a Gaussian stochastic process. Kozachenko et al. (2005) estimated probability that supremum of a stochastic process from Orlicz spaces of random variables exceeds some function. Kozachenko and Moklyachuk (2003) obtained estimates of the distribution of the supremum of stochastic processes from the Orlicz space of random variables.

Kozachenko and Sergiienko (2014) constructed tests for a hypothesis concerning the form of the covariance function of a Gaussian stochastic process. Kozachenko et al. (2005) estimated probability that supremum of a stochastic process from Orlicz spaces of exponential type exceeds some function. Kozachenko and Moklyachuk (2003) obtained estimates of the distribution of the supremum of stochastic processes from the Orlicz space of random variables. Kozachenko and Mlavets (2015) estimated the accuracy and reliability (in $L_p(T)$ metrics) for the calculation of improper integrals depending on a parameter using the Monte Carlo method and the theory of Orlicz subspace $F_\psi(\Omega)$. Yamnenko (2015) studied deviations of stochastic processes from Orlicz spaces of exponential type and obtained a bound for the distributions of norms in the space $L_p(T)$. Yamnenko (2016) obtained an estimate for distributions of norms of deviations of a stochastic process from the Orlicz space from the class $\Delta_2$ of Orlicz functions.

This paper extends and generalizes results of Kozachenko and Moklyachuk (2003) and Yamnenko (2016).

The method of majorizing measures which is applied for proving the main lemma of this paper is extensively used to determine conditions of boundedness and sample path continuity with probability one of Gaussian stochastic processes. In some cases the method of majorizing measures turns out to be more effective than the entropy method which was exploited by Dudley (1973), Nanopoulos and Nobelis (1978), Kôno (1980). More details on application of the method of majorizing measures for obtaining estimates for distributions of stochastic processes can be found in papers by Fernique (1971, 1975), Talagrand (1987, 1996) and by Ledoux and Talagrand (1991), Ledoux (1996).

The paper has the following structure. In section 1 basic definitions from the theory of Orlicz space are given. Section 2 contains general estimates for the distribution of deviations of stochastic processes from Orlicz spaces over some continuous function. Section 3 presents results for random processes from the class $\Delta_2$ and class E. A special case for a $\phi$-sub-Gaussian random process is considered.

### 2. Orlicz Spaces

In this section we summarize the necessary definitions and results about Orlicz spaces of random variables. Krasnoselskii and Rutitskii (1961) give the following definition of an Orlicz N-function.

**Definition 2.1** A continuous even convex function $\{U(x), \ x \in \mathbb{R}\}$ is said to be an Orlicz N-function if it is strictly increasing for $x > 0$, $U(0) = 0$ and

$$\frac{U(x)}{x} \to 0 \text{ as } x \to 0 \text{ and } \frac{U(x)}{x} \to \infty \text{ as } x \to \infty.$$

Any Orlicz N-function $U$ has the following properties

a) $U(\alpha x) \leq \alpha U(x)$ for any $0 \leq \alpha \leq 1$;
b) $U(x) + U(y) \leq U(|x| + |y|)$;

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c) function \( \frac{U(x)}{x} \) increases for \( x > 0 \).

**Example 2.1** The following functions are N-functions.

- \( U(x) = \alpha |x|^\beta, \alpha > 0, \beta > 1; \)
- \( U(x) = \exp{|x|} - |x| - 1; \)
- \( U(x) = \exp{\alpha |x|^\beta} - 1, \alpha > 0, \beta > 1; \)
- \( U(x) = \begin{cases} 
\left( e^{\frac{x^2}{2}} \right)^x, & |x| \leq \left( \frac{1}{\alpha} \right)^{\frac{1}{\beta}}; \\
\exp{\{|x|^\alpha\}}, & |x| > \left( \frac{2}{\alpha} \right)^{\frac{1}{\beta}}, \ 0 < \alpha < 1. 
\end{cases} \)

Let \((\mathbb{T}, \mathcal{B}, \mu)\) be a measurable space with a finite measure \( \mu(\mathbb{T}) < \infty \).

**Definition 2.2** The space \( L_U^\mu(\mathbb{T}) \) of measurable functions on \((\mathbb{T}, \mathcal{B}, \mu)\) such that for any \( f \in L_U^\mu(\mathbb{T}) \) there exists a constant \( r_f \) for which

\[
\int_{\mathbb{T}} U\left( \frac{f(t)}{r_f} \right) d\mu(t) < \infty
\]

is called the Orlicz space.

The space \( L_U^\mu(\mathbb{T}) \) is a Banach space with the Luxemburg norm

\[
\|f\|_{L_U^\mu(\mathbb{T})} = \inf\left\{ r > 0 : \int_{\mathbb{T}} U\left( \frac{f(t)}{r} \right) d\mu(t) \leq 1 \right\}. \tag{1}
\]

We will also consider the Orlicz space \( L_U^{\mu \times \mu}(\mathbb{T} \times \mathbb{T}) \) of measurable functions on \((\mathbb{T} \times \mathbb{T}, \mathcal{B} \times \mathcal{B}, \mu \times \mu)\), where \( \mathcal{B} \times \mathcal{B} \) is the tensor-product sigma-algebra on the product space and \( \mu \times \mu \) is the product measure on the measurable space \((\mathbb{T} \times \mathbb{T}, \mathcal{B} \times \mathcal{B})\). In other words for any \( f \in L_U^{\mu \times \mu}(\mathbb{T} \times \mathbb{T}) \) there exists a constant \( r_f \) for which

\[
\iint_{\mathbb{T} \times \mathbb{T}} U\left( \frac{f(t,s)}{r_f} \right) \ d(\mu(t) \times \mu(s)) < \infty.
\]

**Definition 2.3** Let \( \{U(x), x \in \mathbb{R}\} \) be an Orlicz N-function. The function \( \{U^*(x), x \in \mathbb{R}\} \) for which

\[
U^*(x) = \sup_{y \in \mathbb{R}} (xy - U(y))
\]

is called the Young-Fenchel transform of the function \( U \).

**Remark 2.1** If \( x > 0 \) then

\[
U^*(x) = \sup_{y > 0} (xy - U(y)), \quad U^*(-x) = U^*(x).
\]

**Theorem 2.1** (Krasnoselskii and Rutitskii, 1961) Suppose that \( U \) is an Orlicz N-function. Then

\[
(U^*)^* = U.
\]

Let us give an example of convex conjugate functions.
Example 2.2 Suppose that $p > 1$ and $q$ is the conjugate exponent of $p$, that is $\frac{1}{p} + \frac{1}{q} = 1$. Let $U(x) = \frac{|x|^p}{p}$, then $U^*(x) = \frac{|x|^q}{q}$.

Let $U$ be an Orlicz N-function and $f \in L_U^\mu(\mathbb{T})$. Consider

$$ s(f; U) = \int_{\mathbb{T}} U(f(t))d\mu(t) < \infty. $$

In the space $L_U^\mu(\mathbb{T})$ one can introduce a different norm which is equivalent to the Luxemburg norm. This is the Orlicz norm (Krasnosel’skii and Rutitskii, 1961)

$$ \|f\|_{(U),\mu}^\mu = \sup_{v: s(v; U^*) \leq 1} \int_{\mathbb{T}} f(t)d\mu(t), $$

(2)

where $U^*$ is the Young-Fenchel transform of the function $U$.

Lemma 2.2 (Hölder inequality) Let $\{f(t), t \in \mathbb{T}\}$ be a function from the space $L_U^\mu(\mathbb{T})$ endowed with the Luxemburg norm (1) and let $\{\varphi(t), t \in \mathbb{T}\}$ be a function from the space $L_{(U^*)}(\mathbb{T})$ endowed with the Orlicz norm (2). Then the following inequality holds true

$$ \int_{\mathbb{T}}|f(t)\varphi(t)|d\mu(t) \leq \|f\|_{(U),\mu}^\mu \times \|\varphi\|_{(U^*),\mu}^{U^*}. $$

(3)

Lemma 2.3 (Krasnosel’skii and Rutitskii, 1961) Let $U(x)$ be a $N$-function, $U^*(x)$ be the Young-Fenchel transform of $U(x)$ and let $\chi_A(t)$ be the indicator function of a set $A \subset \mathcal{B}$. Then

$$ \|\chi_A\|_{(U^*),\mu}^{U^*} = \mu(A)U^{-1}\left(\frac{1}{\mu(A)}\right). $$

(4)

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the standard probability space. Kozachenko (1985) gives the following definition of the Orlicz space of random variables, $L_U(\Omega)$.

Definition 2.4 The space $L_U^p(\Omega) = L_U(\Omega)$ of random variables $\xi = \{\xi(\omega), \omega \in \Omega\}$ is called an Orlicz space of random variables, i.e. Orlicz space $L_U(\Omega)$ is the family of random variables where for each $\xi \in L_U(\Omega)$ there exists a constant $r_\xi > 0$ such that

$$ EU\left(\frac{\xi}{r_\xi}\right) < \infty. $$

In this space the Luxemburg norm takes the form

$$ \|\xi\|_{L_U} = \inf\left\{r > 0: EU\left(\frac{\xi}{r}\right) \leq 1\right\}. $$

Example 2.3 Suppose that $U(x) = |x|^p$, $x \in \mathbb{R}$, $p \geq 1$. Then $L_U(\Omega)$ is the well-known space $L_p(\Omega)$ and the Luxemburg norm $\|\xi\|_U$ coincides with the norm

$$ \|\xi\|_p = \left(E |\xi|^p\right)^{\frac{1}{p}}. $$

The following lemma follows from the Chebyshev’s inequality.

Lemma 2.4 (Kozachenko, 1985) Let $\xi$ be a random variable from $L_U(\Omega)$. For any $x > 0$ the next inequality holds
\[ P[|\xi| > x] \leq \left( U\left( \frac{x}{\|\xi\|_U}\right) \right)^{-1}. \]  \tag{5}

**Definition 2.5** Let \( \{X(t), t \in \mathbb{T}\} \) be a random process. The process \( X \) belongs to the Orlicz space \( L_U(\Omega) \) if all random variables \( X(t), t \in \mathbb{T} \) belong to the space \( L_U(\Omega) \) and \( \sup_{t \in \mathbb{T}} \|X(t)\|_U < \infty \).

**Example 2.4** Suppose that there exists a nonnegative function \( c = \{c(t), t \in \mathbb{T}\} \) such that \( P[|X(t)| \leq c(t)] = 1 \) for any \( t \in \mathbb{T} \). Then \( X \) is a \( L_U(\Omega) \)-process for any Orlicz space \( L_U(\Omega) \).

### 3. Distribution of Deviations of Stochastic Processes from Orlicz Spaces

Let \( (\mathbb{T}, \rho) \) be a compact separable metric space equipped with the metric \( \rho \) and let \( \mathcal{B} \) be the Borel sigma-algebra on \( (\mathbb{T}, \rho) \). Let also \( (\mathbb{T}, \mathcal{B}, \mu) \) be a measurable space with a finite measure \( \mu(\mathbb{T}) < \infty \).

Consider a separable stochastic process \( X = \{X(t), t \in \mathbb{T}\} \) from the Orlicz space \( L_U(\Omega) \).

**Assumption \( \Sigma \).** Suppose that \( \sigma = \{\sigma(h), h > 0\} \) is such a continuous function that \( \sigma(h) \geq 0, \sigma(h) \) increases in \( h > 0 \), \( \sigma(h) \to 0 \) as \( h \to 0 \) and
\[
\sup_{\rho(t,s) \leq h} \|X(t) - X(s)\|_U \leq \sigma(h), \quad t, s \in \mathbb{T}.
\]

Note, that since the process \( X \) is continuous in the norm \( \|\cdot\|_U \), then at least one such function exists, for example
\[
\sigma(h) = \sup_{\rho(t,s) \leq h} \|X(t) - X(s)\|_U.
\]

**Example 3.1** Let \( X \) be a generalized \( L_U(\Omega) \)-process of fractional Brownian motion with Hurst index \( H \in (0,1) \), that is, \( X \) is the \( L_U(\Omega) \)-process with stationary increments and covariance function
\[
R_H(t,s) = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}).
\]

Then \( \sigma(h) = h^H \).

Denote by \( \sigma(h) \) the generalized inverse to \( \sigma(h) \) that is
\[
\sigma^{-1}(h) = \sup\{s: \sigma(s) \leq h\}.
\]

Let \( S \) be such a set from \( \mathcal{B} \) that
\[
(\mu \times \mu) \{ (u,v) \in S \times S: \rho(u,v) \neq 0 \} > 0. \tag{6}
\]

Consider a sequence \( \epsilon_k(t) > 0 \) such that \( \epsilon_k(t) > \epsilon_{k+1}(t), \epsilon_k(t) \to 0 \) as \( k \to \infty \) and
\[
\epsilon_1(t) = \sup_{s \in S} \rho(t,s).
\]

Put
\[
C(t,u) = \{ s: \rho(t,s) \leq u \}
\]
and consider two majorizing sequences...
\[ C_{t,k} = C_t(\varepsilon_k(t)) \]

and
\[ \mu_k(t) = \mu(C_{t,k} \cap S). \]

**Assumption F.** Assume that for a continuous function \( f = \{f(t), t \in \mathbb{T}\} \) there exists such a continuous increasing function \( \delta(y) > 0, \ y > 0, \) that \( \delta(y) \to 0 \) as \( y \to 0 \) and the following condition is satisfied
\[ |f(u) - f(v)| \leq \delta(\rho(u, v)), \quad u, v \in \mathbb{T}. \]

Put
\[ d_X(u, v) = \|X(u) - X(v)\|_U \]

and
\[ d_{X,f}(u, v) = \|X(u) - X(v) - f(u) + f(v)\|_U. \]

**Lemma 3.1** Suppose that \( X = \{X(t), t \in \mathbb{T}\} \) is a separable stochastic process from the Orlicz space \( L_U(\Omega) \) which satisfies Assumption \( \Sigma. \) Let \( f \) be a function satisfying Assumption \( F, \) let \( \zeta(y), y > 0 \) be an arbitrary continuous increasing function such that \( \zeta(y) \to 0 \) as \( y \to 0, \) and let
\[ \frac{X(u) - X(v) - f(u) + f(v)}{\zeta(d_f(u, v))} \in L^{\mu \times \mu}_{U}(\mathbb{T} \times \mathbb{T}). \]

Then for any \( S \in \mathcal{B} \) satisfying (6) the next inequality holds true with probability one
\[ \sup_{t \in S} \left| X(t) - f(t) - \frac{1}{\mu(S)} \int_S (X(u) - f(u)) \, d\mu(u) \right| \leq \left\| \frac{X(u) - X(v) - f(u) + f(v)}{\zeta(d_f(u, v))} \right\|_{U, \mu \times \mu}^{S \times S} \times \sup_{t \in S} \sum_{l=1}^\infty \zeta \left( \sum_{k=l}^{l+1} \left( \sigma(\varepsilon_k(t)) + \delta(\varepsilon_k(t)) \right) \right) \, u^{(-1)} \left( \frac{1}{\mu(t) \mu_{l+1}(t)} \right). \]

**Proof:** Let \( V \) be a set of separability of the separable stochastic process \( X \) and consider an arbitrary point \( t \in S \cap V, \) where the set \( S \) is from (6). Put
\[ \tau_l(u) = \frac{\chi_{C_{l,n} \cap S}(u)}{\mu_l(t)}, \]

where \( \chi_A(u) \) is the indicator function on a set \( A. \) Then
\[ \left\| X(t) - f(t) - \int_S (X(u) - f(u)) \tau_l(u) \, d\mu(u) \right\|_U \]
\[ \leq \int_S \left\| (X(t) - X(u)) - (f(t) - f(u)) \right\|_U \tau_l(u) \, d\mu(u) \]
\[ \leq \int_S \|X(t) - X(u)\|_U \tau_l(u) \, d\mu(u) + \int_S |f(t) - f(u)| \tau_l(u) \, d\mu(u) \]
\[ \leq \sigma(\varepsilon_l(t)) + \delta(\varepsilon_l(t)) \to 0 \quad \text{as} \quad l \to \infty. \]

After applying Lemma 2.4, it follows from (8) that
\[ \int_S (X(u) - f(u)) \tau_l(u) \, d\mu(u) \to X(t) - f(t) \]
in probability as \( l \to \infty. \) Then there exists a sequence \( l_n \) such that
with probability one as $l_n \to \infty$. And we obtain the following expansion.

\[
\begin{align*}
|X(t) - f(t) - \int_S (X(u) - f(u)) \tau_t(u) d\mu(u)|
&= |X(t) - f(t) - \int_S (X(u) - f(u)) \tau_t(u) d\mu(u) \\
&\quad + \sum_{l=1}^{n-1} \left( \int_S (X(u) - f(u)) \tau_{t+1}(u) d\mu(u) - \int_S (X(u) - f(u)) \tau_t(u) d\mu(u) \right) \\
&\leq |X(t) - f(t) - \int_S (X(u) - f(u)) \tau_t(u) d\mu(u) \\
&\quad + \sum_{l=1}^{n-1} \left| \int_S (X(u) - f(u)) \tau_{t+1}(u) d\mu(u) - \int_S (X(u) - f(u)) \tau_t(u) d\mu(u) \right|.
\end{align*}
\]

(9)

Using (9) we obtain with probability one the following inequality

\[
\begin{align*}
|X(t) - f(t) - \frac{1}{\mu(S)} \int_S (X(u) - f(u)) \, d\mu(u)|
&= \left| \sum_{l=1}^{\infty} \int_S (X(u) - f(u)) \tau_{t+1}(u) d\mu(u) - \int_S (X(u) - f(u)) \tau_t(u) d\mu(u) \right| \\
&\leq \sum_{l=1}^{\infty} \left| \int_S (X(u) - f(u)) \tau_{t+1}(u) d\mu(u) - \int_S (X(u) - f(u)) \tau_t(u) d\mu(u) \right| \\
&= \sum_{l=1}^{\infty} \left| \int_S (X(u) - X(v) - f(u) + f(v)) \tau_{t+1}(u) \tau_t(v) d\mu(u) d\mu(v) \right| \\
&\leq \int_{S \times S} \frac{|X(u) - X(v) - f(u) + f(v)|}{\zeta \left( d_f(u, v) \right)} \left( \sum_{l=1}^{\infty} \tau_{t+1}(u) \tau_t(v) \zeta \left( d_f(u, v) \right) \right) \times d\left( \mu(u) \times \mu(v) \right).
\end{align*}
\]

(10)

From the H"older inequality and (10) the following inequality holds true

\[
\begin{align*}
|X(t) - f(t) - \frac{1}{\mu(S)} \int_S (X(u) - f(u)) \, d\mu(u)| &\leq \left| \frac{|X(u) - X(v) - f(u) + f(v)|}{\zeta \left( d_f(u, v) \right)} \right|_{S \times S}^{\mu, \mu} \\
&\quad \left( \sum_{l=1}^{\infty} \tau_{t+1}(u) \tau_t(v) \zeta \left( d_f(u, v) \right) \right) \left( d_f(u, v) \right)_{\mu, \mu}. 
\end{align*}
\]

(11)

From the Assumption F and the following relationship
\[
d_f(u, t) = \|X(u) - X(v) - f(u) + f(v)\|_U \leq d_X(u, v) + |f(u) - f(v)| \leq d_X(u, v) + \delta(\rho(u, v)),
\]

we have
\[
\begin{align*}
\tau_{t+1}(u) \tau_t(u) \zeta \left( d_f(u, v) \right) &\leq \tau_{t+1}(u) \tau_t(u) \zeta \left( d_f(u, t) \right) + \delta(\tau_t(u)) \\
&\leq \tau_{t+1}(u) \tau_t(u) \zeta \left( \sigma(t) + \delta(t) \right) + \sigma(t) + \delta(t).
\end{align*}
\]

(12)
From (11) and (12) we obtain the next inequality
\[ \left| X(t) - f(t) - \frac{1}{\mu(S)} \int_S (X(u) - f(u)) d\mu(u) \right| \leq \left\| \frac{X(u) - X(v) - f(u) + f(v)}{\zeta(d_f(u,v))} \right\|_{\|,\mu \times \mu}^{S \times S} \sum_{t=1}^{\infty} \sum_{k=1}^{l} \left( \sum_{k=1}^{l} (\sigma(\epsilon_k(t)) + \delta(\epsilon_k(t))) \right) \]
\[ \| \tau_{l+1}(u) \tau_{l}(v) \|_{(u',\mu \times \mu)}^{S \times S} \]
\[ = \left( \frac{1}{\mu(t)\mu_{l+1}+1(t)} \right) \sup_{(u',v')} \int_S \left( X(u) - f(u) \right) d\mu(u) \]
\[ \leq \left( \frac{1}{\mu(t)\mu_{l+1}+1(t)} \right) \sup_{(u',v')} \int_S \left( X(u) - f(u) \right) d\mu(u) \]
(13)

Since \( V \) is a set of separability of the process \( X \) and \( t \in S \cap V \), then \( S \cap V \) is the countable set and (14) holds true with probability one for all \( t \in S \cap V \). Therefore
\[ \sup_{t \in S} \left| X(t) - f(t) - \frac{1}{\mu(S)} \int_S (X(u) - f(u)) d\mu(u) \right| = \sup_{t \in S \cap V} \left| X(t) - f(t) - \frac{1}{\mu(S)} \int_S (X(u) - f(u)) d\mu(u) \right| \]
with probability one.

**Corollary 3.2** Suppose that assumptions of Lemma 3.1 are satisfied. Put
\[ \zeta_1(t) = \zeta(2 \max\{\sigma(\epsilon_1(t)) + \sigma(\epsilon_2(t)), \delta(\epsilon_1(t)) + \delta(\epsilon_2(t))\}) \]
and
\[ \nu_t(u) = \mu \left( C_t \left( \sigma^{-1}\left( \frac{\zeta_1^{-1}(u)}{2} \right) \right) \cap S \right). \]

Then for any \( 0 < p < 1 \) the next inequality holds true
\[ \sup_{t \in S} \left| X(t) - f(t) - \frac{1}{\mu(S)} \int_S (X(u) - f(u)) d\mu(u) \right| \leq \eta_f C_p \]
(15)
with probability one, where
\[ \eta_f = \left\| \frac{X(u) - X(v) - f(u) + f(v)}{\zeta(d_f(u,v))} \right\|_{\|,\mu \times \mu}^{S \times S} \]
\[ = \left( \frac{1}{\mu(t)\mu_{l+1}+1(t)} \right) \sup_{(u',v')} \int_S \left( X(u) - f(u) \right) d\mu(u) \]
(16)
and
\[ C_p = \sup_{t \in S} \frac{1}{p(1-p)} \int_0^{\zeta_1(t)} U^{(1)} \left( \frac{1}{\nu_t(u)/\nu_t(v)} \right) du. \]
(17)

**Proof:** Let the sequence \( \epsilon_k(t), \ k \geq 1 \), be defined as follows
\[ \epsilon_1(t) = \sup_{s \in S} \rho(t,s), \quad \epsilon_k(t) = \sigma^{-1}\left( \zeta_1^{-1}\left( \zeta_1(t)p^{k-1} \right) \right). \]

Then
\[ \zeta(\max\{\sigma(\epsilon_1(t)) + \sigma(\epsilon_{l+1}(t)), \delta(\epsilon_1(t)) + \delta(\epsilon_{l+1}(t))\}) \leq \zeta_1(t)p^{l-1}, \]
\[ \mu(t) = \mu(C_t(\epsilon_1(t)) \cap S) = \nu_t(\zeta_1(t)p^{l-1}). \]
and
Proof: Using the Fubini’s theorem and (18) we obtain that with probability one
The assertion of the theorem follows from (16). Thus

\[ \int_S \int_S U \left( \frac{X(u) - X(v) - f(u) + f(v)}{\zeta \left( d_f(u, v) \right) r} \right) d(\mu(u) \times \mu(v)) \]

\[ \leq \int_S \int_S U \left( \frac{|X(u) - X(v)| + |f(u) - f(v)|}{\zeta \left( d_f(u, v) \right) r} \right) d(\mu(u) \times \mu(v)) < \infty, \]

and therefore the process

\[ \frac{X(u) - X(v) - f(u) + f(v)}{\zeta \left( d_f(u, v) \right)} \]

with probability one belongs to the space \( L^{u \times \mu}_{S \times S} \). Thus

\[ \eta_f = \left| \frac{X(u) - X(v) - f(u) + f(v)}{\zeta \left( d_f(u, v) \right)} \right|_{u \mu \times \mu}^{S \times S}, \]

is a finite with probability one random variable. It follows from (15) that

\[ \sup_{t \in S} |X(t) - f(t)| \leq \frac{1}{\mu(S)} \int_S (X(u) - f(u)) \, d\mu(u) + \eta_f C_p \] (20)

with probability one. Since \( X(u) \in L_U(\Omega) \) for any \( u \in S \) then \( X(u) - f(u) \in L_U(\Omega) \) as well and

\[ \frac{1}{\mu(S)} \int_S (X(u) - f(u)) \, d\mu(u) \in L_U(\Omega). \]

Therefore

\[ \left\| \frac{1}{\mu(S)} \int_S (X(u) - f(u)) \, d\mu(u) \right\|_U \leq \frac{1}{\mu(S)} \int_S \| X(u) - f(u) \|_U \, d\mu(u) \leq \sup_{u \in S} \| X(u) - f(u) \|_U < \infty. \]

It follows from Lemma 2.4 that for any \( y > 0 \)

\[ P \left\{ \left\| \frac{1}{\mu(S)} \int_S (X(u) - f(u)) \, d\mu(u) \|_U > y \right\} \right\} \leq \frac{1}{\mu(S)} \left( \frac{y}{\int_S (X(u) - f(u)) \, d\mu(u)} \right). \] (21)

Using (20) we obtain that for any \( 0 \leq \alpha \leq 1 \) and \( x > 0 \)

\[ P \left\{ \sup_{t \in S} |X(t) - f(t)| > x \right\} \leq P \left\{ \frac{1}{\mu(S)} \int_S (X(u) - f(u)) \, d\mu(u) \right\|_U > ax \} + P \left\{ \eta_f C_p > (1 - \alpha)x \right\}. \] (22)

The assertion of the theorem follows from (21) and (22).

\[ \square \]

4. Distribution of Deviations of Stochastic Processes from the Class \( \Delta^2 \)

Krasnoselskii and Rutitskii (1961) introduce the following class of Orlicz N-functions.

**Definition 4.1** A N-function \( U(x) \) belongs to the class \( \Delta^2 \) if there exist such constants \( x_0 \geq 0 \) and \( L > 1 \) such that

\[ U^2(x) \leq U(Lx) \quad \text{for} \quad x \geq x_0. \]
Example 4.1

a) For any $\alpha \geq 1$ and $c > 0$ the function $U(x) = \exp\left(\frac{c}{x^\alpha}\right) - 1$ belongs to the class $\Delta^2$ with $x_0 = 0$ and $L = 2^\frac{1}{\alpha}$.
b) The function $U(x) = \exp(\phi(x) - 1)$, where $\phi(x)$ is an Orlicz $N$-function belongs to $\Delta^2$ with $x_0 = 0$ and $L = 2$.

Lemma 4.1 Let $U$ be a function from the class $\Delta^2$. Then for all $x \geq U(x_0)$ the next inequality holds true

$$U^{(-1)}(x^2) \leq L U^{(-1)}(x).$$

Definition 4.2 A stochastic process $X = \{X(t), t \in \mathbb{T}\}$ belongs to the class $\Delta^2$ if $X \in L_U(\Omega)$ and $U$ is an Orlicz function from the class $\Delta^2$.

Theorem 4.2 Suppose that $X = \{X(t), t \in \mathbb{T}\}$ is a separable stochastic process from the class $\Delta^2$ which satisfies Assumption $\Sigma$ and $f = \{f(t), t \in \mathbb{T}\}$ be a function satisfying Assumption $F$. Let $\gamma = \{\gamma(u), u > 0\}$ be arbitrary increasing continuous function such that the function $\zeta(u) = \gamma(u) / u$ is increasing and $\zeta(\gamma) \to 0$ as $y \to 0$, and let

$$\frac{X(u) - X(v) - f(u) + f(v)}{\zeta(u)} \nu_f(u,v) \in L_U^\mu \times \mu (\mathbb{T} \times \mathbb{T}).$$

Suppose also that

$$\sup_{t \in \mathbb{T}} \int_0^{\zeta_1(t)} U^{(-1)} \left( \frac{1}{\nu_f(u)} \right) du < \infty,$$

(23)

where $\zeta_1(t)$ and $\nu_f(u)$ are defined in Corollary 3.2. Then for any $0 < p < 1$ the next inequality holds true with probability one

$$\sup_{t \in \mathbb{T}} \left| X(t) - f(t) - \int_S \left( X(u) - f(u) \right) \frac{d\mu(u)}{\mu(S)} \right| \leq \frac{\eta_f}{p(1-p)} \sup_{t \in S} \left( L \int_0^{\min(\zeta_1 p, r_0)} U^{(-1)} \left( \frac{1}{\nu_f(u)} \right) du + Z(t) \right) = \eta_f a_p,$$

(24)

where

$$\eta_f = \left\| \left( \frac{X(u) - X(v) - f(u) + f(v)}{\nu_f(u,v)} \right) \right\|_{L_U^\mu \times \mu} \times S \times S$$

(25)

is a finite with probability one random variable, $r_0$ is such a number that $\left( \nu_f(u) \right)^{-1} \geq U(x_0)$, if $u \leq r_0; Z(t) = 0$ if $\zeta_1(t)p \leq r_0$ and

$$Z(t) = \int_{\min(\zeta_1(t)p, r_0)}^{\zeta_1(t)p} U^{(-1)} \left( \frac{1}{\nu_f \left( \frac{u}{p} \right) \nu_f(u)} \right) du$$

if $\zeta_1(t)p > r_0$.

Proof: This theorem follows from the Corollary 3.2. Indeed, Lemma 4.2 implies that

$$U^{(-1)} \left( \frac{1}{\nu_f \left( \frac{u}{p} \right) \nu_f(u)} \right) \leq U^{(-1)} \left( \left( \nu_f(u) \right)^2 \right) \leq L U^{(-1)} \left( \left( \nu_f(u) \right)^{-1} \right)$$

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if $(v_t(u))^{-1} \geq U(x_0)$. Therefore if (23) is satisfied then
\[
\sup_{t \in S} \frac{1}{p(1 - p)} \int_0^{\zeta_1(t)p} U^{-1} \left( \frac{1}{v_t(u)} \right) \frac{d u}{v_t(u)} < a_p.
\]
Next, if we put
\[
\bar{\gamma} = \sup_{u, v \in S} \gamma(d(u, v)) < \infty,
\]
then
\[
\eta_f = \bar{\gamma} \left\| \frac{X(u) - X(v) - f(u) + f(v)}{d_f(u, v)} \right\|_{\mu \times \mu} \leq \bar{\gamma} \left\| \frac{X(u) - X(v) - f(u) + f(v)}{d_f(u, v)} \right\|_{\mu \times \mu}.
\]

A random variable
\[
\frac{X(u) - X(v) - f(u) + f(v)}{d_f(u, v)}
\]
is from the space $L_0^{\phi}(S \times S)$ with probability one. Since
\[
E \int_S \int_S U \left( \frac{X(u) - X(v) - f(u) + f(v)}{d_f(u, v)} \right) d(\mu(u) \times \mu(v)) \leq \int_S \int_S E U \left( \frac{X(u) - X(v) - f(u) + f(v)}{d_f(u, v)} \right) d(\mu(u) \times \mu(v)) \leq \mu(S)^2,
\]
then $\eta_f < \infty$ with probability one.

Consider now applications of this theorem for processes from the space $L_0(\Omega)$ with $U(x) = \exp\{\phi(x)\} - 1$, where $\phi(x)$ is an Orlicz N-function from the class E. Such Orlicz spaces of exponential type is also known as $\text{Sub}_\phi(\Omega)$, the space of $\phi$-sub-Gaussian random variables. This space is more general than the class of Gaussian random variables (see Buldygin and Kozachenko (2000), Kozachenko et al. (2005) for details).

Kozachenko (1985) gives the definition of the class E of Orlicz N-functions. Recall that an Orlicz N-function $U \in E$ if there exist constants $z_0 \geq 0, B > 0, D > 0$ such that for all $x \geq z_0, y \geq z_0$ the following inequality holds true
\[
U(x)U(y) \leq B U(Dxy).
\]

**Example 4.2**

a) Let $U(x) = c|x|^p, c > 0, p > 1$, then $U$ belongs to the class E with constants $B = c, z_0 = 0$ and $D = 1$.

b) The function $U(x) = |x|^\beta / (\log(c + |x|))^\alpha$ belongs to the class E if $c$ is a number large enough that the function $U(x)$ be convex. In this case $z_0 = \max\{0, \exp\left(2^{-\frac{1}{\beta}}\right) - c\}.$

**Lemma 4.3 (Buldygin and Kozachenko, 2000)** Each function from the class $\Delta^2$ also belongs to the class $E$. 

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For simplicity consider the case where \( z_0 = 0 \).

**Lemma 4.4** Let all assumptions of Theorem 4.2 be satisfied for the Orlicz \( N \)-function \( U(x) = \exp\{\phi(x)\} - 1 \) where \( \phi(x) \) is an \( N \)-function from the class \( E \). Let there exists an increasing function \( \psi = \{\psi(\alpha), 0 \leq \alpha \leq 1\} \) such that \( 0 \leq \psi(\alpha) \leq 1 \) and \( \phi(\alpha x) \leq \psi(\alpha) \phi(x) \) for all \( 0 \leq \alpha \leq 1 \). Then for all \( x \geq D\tilde{\gamma}\phi^{-1}(B) \) where \( B \) and \( D \) are from Definition 4.1 the following inequality holds true

\[
P(\eta_f > x) \leq 2 \exp \left(- \left( \log \left( 1 + \frac{1}{A(S)} \right) \right) \frac{1}{B} \phi \left( \frac{x}{\tilde{\gamma}} \right) \right)
\]

(26)

where \( \tilde{\gamma} = \sup_{u,v \in S} \gamma \left( d_f(u,v) \right) \), the random variable \( \eta_f \) is determined in (25) and

\[
A(S) = \int_{S} \int_{S} \psi \left( \frac{\gamma(d_f(u,v))}{\tilde{\gamma}} \right) \, d(\mu(u) \times \mu(v)).
\]

**Proof:** It is not difficult to obtain that

\[
P(\eta_f > x)
\]

\[
\leq P \left( \int_{S} \int_{S} \exp \left\{ \phi \left( \frac{X(u) - X(v) - f(u) + f(v)\gamma(d_f(u,v))}{d_f(u,v)x} \right) \right\} - 1 \right) \, d(\mu(u) \times \mu(v)) > 1 \). (27)

Since

\[
\exp \left\{ \phi \left( \frac{X(u) - X(v) - f(u) + f(v)\gamma(d_f(u,v))}{d_f(u,v)x} \right) \right\} - 1 \leq \psi \left( \frac{\gamma(d_f(u,v))}{\tilde{\gamma}} \right) \left( \exp \left\{ \phi \left( \frac{X(u) - X(v) - f(u) + f(v)\gamma}{d_f(u,v)x} \right) \right\} - 1 \right),
\]

then for any \( p > 1 \) the next relations take place

\[
P(\eta_f > x)
\]

\[
\leq P \left( \int_{S} \int_{S} \psi \left( \frac{\gamma(d_f(u,v))}{\tilde{\gamma}} \right) \left( \exp \left\{ \phi \left( \frac{X(u) - X(v) - f(u) + f(v)\gamma}{d_f(u,v)x} \right) \right\} - 1 \right) \, d(\mu(u) \times \mu(v)) \right)^p
\]

\[
> 1 + A(S)
\]

\[
\leq E \left( \frac{A(S)}{1 + A(S)} \right)^p \int_{S} \int_{S} \psi \left( \frac{\gamma(d_f(u,v))}{\tilde{\gamma}} \right) \left( \exp \left\{ p \phi \left( \frac{X(u) - X(v) - f(u) + f(v)\gamma}{d_f(u,v)x} \right) \right\} - 1 \right) \, d(\mu(u) \times \mu(v))
\]

\[
\times \frac{d(\mu(u) \times \mu(v))}{A(S)}.
\]

(28)

From the other hand since \( \phi \) is from the class \( E \)

\[
p\phi \left( \frac{X(u) - X(v) - f(u) + f(v)\gamma}{d_f(u,v)x} \right) = \frac{1}{B} \phi \left( \phi^{-1}(pB) \right) \phi \left( \frac{X(u) - X(v) - f(u) + f(v)\gamma}{d_f(u,v)x} \right)
\]

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\[ \phi \left( \frac{d \phi^{-1}(pB) \left( X(u) - X(v) \right)}{d \left( f(u), v \right)} \right) \leq \phi \left( \frac{d \phi^{-1}(pB) \left( X(u) - X(v) \right)}{d \left( f(u), v \right)} \right). \] (29)

Let's put \( p = \frac{1}{B} \phi \left( \frac{x}{D_f} \right) \) in (28) then for \( \phi \left( \frac{x}{D_f} \right) > B \) it follows from (28) and (29) that
\[
P(\eta_f > x)
\leq \left( \frac{A(S)}{1 + A(S)} \right) \frac{\phi \left( \frac{x}{D_f} \right)}{B} \int_S \psi \left( \gamma \left( \frac{d f(u), v}{\bar{\gamma}} \right) \right) E \exp \left( \phi \left( \frac{X(u) - X(v) - f(u) + f(v)}{d \left( f(u), v \right)} \right) \right) \frac{d \left( \mu(u) \times \mu(v) \right)}{A(S)} \leq 2 \left( \frac{A(S)}{1 + A(S)} \right) \frac{\phi \left( \frac{x}{D_f} \right)}{B}. \]

\[ \text{Corollary 4.5 Let all assumptions of Lemma 4.4 be satisfied. In this case condition (23) of Theorem 4.2 is of the form} \]
\[
\sup_{t \in S} \int_0^{\zeta(t)} \phi^{-1} \left( \log \left( 1 + \frac{1}{\nu_t(u)} \right) \right) du < \infty \] (30)
and for
\[ x > \phi^{-1}(B) \left( \tilde{a}_p D \bar{\gamma} + \left\| \int_S (X(u) - f(u)) \frac{d \mu(u)}{\mu(S)} \right\|_U \right) \]
the next inequality holds true
\[
P \left( \sup_{t \in S} |X(t) - f(t)| > x \right) \leq 2 \left( \exp \{ - \delta_1 \phi(x^2) \} + \exp \{ - \phi(x^2) \} \right), \] (31)
where
\[ \delta_2 = \left( \tilde{a}_p D \bar{\gamma} + \left\| \int_S (X(u) - f(u)) \frac{d \mu(u)}{\mu(S)} \right\|_U \right), \]
\[ \delta_1 = \frac{1}{B} \log \left( 1 + \frac{1}{A(S)} \right), \]
\[ \tilde{a}_p = \frac{1}{p(1 - p)} \sup_{t \in S} \left( L \int_0^{\zeta(t)p} \phi^{-1} \left( \log \left( 1 + \frac{1}{\nu_t(u)} \right) \right) du \right), \]
\[ L = 2 \text{ or } L = 2^{\frac{1}{1 - \alpha}} \text{ if } \phi(x) = |x|^x, \alpha > 1, \text{ and } U(x) = \exp \{ \phi(x) \} - 1. \]

Proof: It is easy to see that \( a_p = \tilde{a}_p \) for the Orlicz N-function \( U(x) = \exp \{ \phi(x) \} - 1 \). It follows from (26) that for any \( 0 \leq \alpha < 1 \) and \( x \geq a_p D \bar{\gamma} \frac{\phi^{-1}(B)}{1 - \alpha} \) the next inequality holds true
\[
P \left( \sup_{t \in S} |X(t) - f(t)| > x \right) \leq \left( \left\| \int_S (X(u) - f(u)) \frac{d \mu(u)}{\mu(S)} \right\|_U > \alpha x \right) \]
\[ + 2 \exp \left\{ - \delta_1 \phi \left( \frac{1 - \alpha}{\tilde{a}_p D \bar{\gamma}} \right) \right\}. \] (32)

Since for any random variable \( \xi \in L_U(\Omega) \) with \( U(x) = \exp \{ \phi(x) \} - 1 \)
\[ \frac{E \exp \left( \frac{\xi}{\| \xi \|_U} \right)}{\exp \{ \phi \left( \frac{\xi}{\| \xi \|_U} \right) \} \} \leq 2 \exp \left\{ - \phi \left( \frac{x}{\| x \|_U} \right) \right\} \]
(33)
then (31) follows from (32) after putting

\[ \alpha = \frac{\left\| \int_{S} (X(u) - f(u)) \frac{d\mu(u)}{\mu(S)} \right\|_U}{\tilde{a}_p D \tilde{y} + \left\| \int_{S} (X(u) - f(u)) \frac{d\mu(u)}{\mu(S)} \right\|_U}. \]

\[ \square \]

**Example 4.3** Let \( L_U(\Omega) \) be the space of sub-Gaussian random variables \( \text{Sub}(\Omega) \) that is \( U(x) = \exp(|x|^2} \) \( - 1 \). In this case we have \( B = D = 1, L = \sqrt{2} \),

\[ \tilde{a}_p \frac{1}{p(1-p)} \sup_{t \in S} \left( \sqrt{2} \int_0^{\xi_1(t)} \log \left( \frac{1}{v_t(u)} + 1 \right)^{\frac{1}{p}} du \right) \]

and condition (30) is satisfied if

\[ \sup_{t \in S} \int_0^{\xi_1(t)p} \left| \log(v_t(u)) \right|^{\frac{1}{r}} du < \infty. \]

If \( \gamma(u) = 1 \) and the metrics \( \rho(u, v) = d(u, v) \) then condition (34) holds true if

\[ \sup_{t \in S} \int_0^{\xi_1(t)} \left| \log(v_t(u)) \right|^{\frac{1}{r}} du < \infty. \]

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**References**


