A Family of Efficient Estimator in Circular Systematic Sampling

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Received 18 August 2014; Published online 13 September 2014

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Abstract

This paper proposes a family of exponential estimators for estimating the population mean $\bar{Y}$ of study variable $y$ using an auxiliary variable $x$ in circular systematic sampling design under single and two phase sampling. The expression of the bias and mean square error of proposed class of estimators are derived in general form. It has been shown that the proposed class of estimators are more efficient than ratio, product, regression and other estimators considered here in circular systematic sampling under single and two phase sampling. An empirical study is carried out in support of theoretical study.

Keywords: Circular systematic sampling; Efficiency; Ratio estimator; Regression estimator; Two-phase sampling

1. Introduction

In sample surveys, the utilization of auxiliary information is frequently acknowledged to higher the accuracy of the estimation of population characteristics under study. The auxiliary data might either be promptly accessible or may be made accessible without much trouble by occupy a part of the survey resources or available from previous experience, census or administrative databases. It is well known that when the auxiliary information is to be used at the estimation stage, the ratio, product and regression methods are widely employed. Many authors including Singh et al. (2007), Shabbir and Gupta (2007), Singh and Kumar (2011), and Sharma and Singh (2013, 2014) suggested estimators using auxiliary variable.

The usual systematic sampling design is quite simple and most commonly used in sample survey. Systematic sampling has an advantage of selecting the whole sample with just one random start. In

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this method of sampling, the first unit is selected randomly and remaining units are selected automatically according to some predetermined patterns. Hasel (1942) and Griffth (1945-1946) found systematic sampling to be efficient and convenient in sampling certain natural populations like forest areas for estimating the volume of the timber and area under different types of cover. Cochran (1946) and Hajec (1959) had stated that in large-scale sampling work, this procedure provides more efficient estimators than those provided by simple random sampling and/or stratified random sampling under certain conditions.

In case of known population mean, $\bar{X}$ of the auxiliary variable $x$, Swain (1964) and Shukla (1971) have suggested the ratio and product estimators for the population mean $\bar{Y}$ of the survey variable $y$, respectively, along with their properties in systematic sampling. Some other remarkable work in this area are Singh and Singh (1998), Singh et al. (2012), Verma et al. (2012), Singh and Jatwa (2012), Singh and Solanki (2012), and Verma et al. (2014).

In linear systematic sampling, given a sample size $n$, sampling is possible only if population size $N$ is divisible by $n$. Even when this condition is satisfied, the scheme cannot provide estimate of variance of the sample mean. This scheme has two drawbacks namely, given $N$, $n$ has limited choice and variance of the sample mean is not estimable. The first limitation could be removed through circular systematic sampling (CSS) as suggested by Lahiri (1951). The procedure consists in selecting a unit, by a random start, from 1 to $N$ and then thereafter selecting every $k^\text{th}$ unit, $k$ being an integer nearest to $N/n$, in a circular manner, until a sample of $n$ units is obtained. Suppose that a unit with random number $i$ is selected. The sample will then consists of the units corresponding to the serial numbers

$$\text{Label} = \begin{cases} 
   i + jk, & 1 \leq i + jk \leq N, \\
   i + jk - N, & N < i + jk.
\end{cases}$$

(for details see Singh and Chaudhary (1986), pp83)

In the following manner, we may draw $N$ circular systematic samples, each of size $n$ as displayed in table 1.

**Table 1 Possible Samples using Circular Systematic Sampling**

<table>
<thead>
<tr>
<th>Sample number</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>i</th>
<th>......</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>$u_2$</td>
<td>$u_i$</td>
<td></td>
<td>$u_N$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u_{k+1}$</td>
<td>$u_{k+2}$</td>
<td>$u_{k+i}$</td>
<td>$u_{2N}$</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>.</td>
<td>.</td>
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<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td></td>
<td>.</td>
</tr>
<tr>
<td>$u_{(n-1)k+1}$</td>
<td>$u_{(n-1)k+2}$</td>
<td>$u_{(n-1)k+i}$</td>
<td>$u_{nN}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
From this $N$ possible sample, a sample of size $n$ is selected randomly to observe $Y$ and $X$.

### 2. Terminology used in Circular Systematic Sampling

Let us suppose that $U^*$ be a finite population consists of $N$ distinct labelled units i.e. $U^* = \{U_1, U_2, \ldots, U_N\}$ and $n$ be a fixed sample size.

Also, let $Y$ and $X$ be study and auxiliary variables taking values $y_{ij}$ and $x_{ij}$, $i = (1,2,\ldots, N), j = (1,2,\ldots, n)$.

The CSS sample means $\bar{Y}_{CSS} = \frac{1}{n} \sum_{j=1}^{n} y_{ij}/n$ and $\bar{X}_{CSS} = \frac{1}{n} \sum_{j=1}^{n} x_{ij}/n$ are unbiased estimates of population means $\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} y_{ij}$ and $\bar{X} = \frac{1}{N} \sum_{j=1}^{N} x_{ij}$ respectively.

The variance of $\bar{Y}_{CSS}$ and $\bar{X}_{CSS}$ under CSS design is written as-

$$V(\bar{Y}_{CSS}) = \theta_1 \left[ 1 + (n-1)\rho_y \right] \frac{S_y^2}{n} = \bar{Y}^2 \tilde{C}_y^2$$

and

$$V(\bar{X}_{CSS}) = \theta_1 \left[ 1 + (n-1)\rho_x \right] \frac{S_x^2}{n} = \bar{X}^2 \tilde{C}_x^2$$

where

$$\theta_1 = \left( \frac{N-1}{N} \right), \quad S_y^2 = \frac{\sum_{i=1}^{N} \sum_{j=1}^{n} (y_{ij} - \bar{Y})^2}{n(N-1)}, \quad S_x^2 = \frac{\sum_{i=1}^{N} \sum_{j=1}^{n} (x_{ij} - \bar{X})^2}{n(N-1)}$$

with

$$\rho_y = \frac{2}{n(n-1)(N-1)S_y^2} \sum_{i=1}^{N} \sum_{j=iu} \left( y_{ij} - \bar{Y} \right) \left( y_{iu} - \bar{Y} \right)$$

and

$$\rho_x = \frac{2}{n(n-1)(N-1)S_x^2} \sum_{i=1}^{N} \sum_{j=iu} \left( x_{ij} - \bar{X} \right) \left( x_{iu} - \bar{X} \right)$$
where \((\rho_y, \rho_x)\) represents intraclass correlation coefficients between pairs of units within the CSS for Y and X, respectively.

also,

\[
\text{Cov}(\overline{Y}_{CSS}, \overline{X}_{CSS}) = \sigma_1 \left[1 + (n-1)\rho_y \right]^{1/2} \left[1 + (n-1)\rho_x \right]^{1/2} \frac{S_{yx}}{n} = \overline{S}_{yx} = \overline{Y}\overline{X}\overline{C}_{yx}
\]

where

\[
S_{yx} = \frac{\sum_{i=1}^{N} \sum_{j=1}^{n} (y_{ij} - \overline{Y})(x_{ij} - \overline{X})}{n(N-1)}
\]

3. Estimators in Literature

In literature, generally, we use ratio, product and regression estimators for estimating the population mean when we have information on auxiliary variables. Thus, we consider ratio, product and regression estimators based on CSS as standard result for making comparison with our suggested class of estimators.

The ratio estimator of the population mean \(\overline{Y}\) based on CSS with known \(\overline{X}\) is defined as

\[
\overline{Y}_R = \overline{Y}_{CSS} \frac{\overline{X}}{\overline{X}_{CSS}}
\]

The product estimator of the population mean \(\overline{Y}\) based on CSS with known \(\overline{X}\) is defined as

\[
\overline{Y}_P = \overline{Y}_{CSS} \frac{\overline{X}_{CSS}}{\overline{X}}
\]

The linear regression estimator of the population mean \(\overline{Y}\) based on CSS with known \(\overline{X}\) is defined as

\[
\overline{Y}_{lr} = \overline{Y}_{CSS} + \hat{\beta}_{yx}(\overline{X} - \overline{X}_{CSS})
\]

where \(\hat{\beta}_{yx} = \frac{S_{yx}}{s_x^2}\) is an estimator for population regression coefficient \(\beta_{yx}\) with

\[
s_x^2 = \frac{\sum_{i=1}^{n} (x_{ij} - \overline{X}_{CSS})^2}{(n-1)} \quad \text{and} \quad S_{yx} = \frac{\sum_{i=1}^{n} (y_{ij} - \overline{X}_{CSS})(x_{ij} - \overline{Y}_{CSS})}{(n-1)}
\]
The bias expressions of estimators \( \bar{y}_R \), \( \bar{y}_P \) and \( \bar{y}_{lr} \) are given as

\[
\text{Bias}(\bar{y}_R) = \bar{y} \left[ 1 - \frac{1}{2} \frac{\text{C}_x^2}{\text{C}_{xy}} \right] \quad (6)
\]

\[
\text{Bias}(\bar{y}_P) = \bar{y} \frac{\text{C}_{xy}}{\text{C}_x} \quad (7)
\]

\[
\text{Bias}(\bar{y}_{lr}) = 0 \quad (8)
\]

and the MSE expressions of estimators \( \bar{y}_R \), \( \bar{y}_P \) and \( \bar{y}_{lr} \) are given as

\[
\text{MSE}(\bar{y}_R) = \bar{y}^2 \left( \frac{\text{C}_{xy}^2}{\text{C}_x^2} + \frac{\text{C}_{xy}^2}{\text{C}_y^2} - 2 \frac{\text{C}_{xy}^2}{\text{C}_x \text{C}_y} \right) \quad (9)
\]

\[
\text{MSE}(\bar{y}_P) = \bar{y}^2 \left( \frac{\text{C}_{xy}^2}{\text{C}_x^2} + \frac{\text{C}_{xy}^2}{\text{C}_y^2} + 2 \frac{\text{C}_{xy}^2}{\text{C}_x \text{C}_y} \right) \quad (10)
\]

\[
\text{MSE}(\bar{y}_{lr}) = \bar{y}^2 \frac{\text{C}_{xy}^2}{\text{C}_y^2} \left( 1 - \frac{\text{C}_{xy}^2}{\text{C}_y^2} \right) \quad (11)
\]

where, \( \frac{\text{C}_{xy}^2}{\text{C}_y^2} \) and \( \text{C}_x^2 \) are given as

\[
\text{C}_x^2 = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2}{mn}
\]

Double sampling scheme is applicable when population mean of auxiliary variable \( \bar{x} \) is unknown. Under double sampling scheme, first we divide the population into \( N \) clusters of size \( n \), each according to CSS, and select randomly \( m \) distinct clusters \((1 < m < k)\) to estimate \( \bar{x} \) only. In second phase, a cluster is selected randomly from \( m \) CSSs to estimate \( \bar{y} \). Hence, the expressions for \( \bar{y}_R \), \( \bar{y}_P \) and \( \bar{y}_{lr} \), with unknown \( \bar{x} \), are given as

\[
\bar{y}_R = \bar{y}_{CSS} \frac{\bar{x}}{\bar{x}_{CSS}}
\]

\[
\bar{y}_P = \bar{y}_{CSS} \frac{\bar{x}_{CSS}}{\bar{x}}
\]

\[
\bar{y}_{lr} = \bar{y}_{CSS} + \hat{\beta}_{xy} (\bar{x}_{CSS} - \bar{x}_{CSS})
\]

where \( \bar{x}_{CSS} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}}{mn} \)
Defined,

\[ V(\bar{X}) = \text{Cov}(\bar{X}_{\text{CSS}}, \bar{X}_{\text{CSS}}) = \theta_1 \left[ 1 + (n-1)\rho_x \right] \frac{S_x^2}{nm} = \frac{\tilde{s}_x^2}{m} \]  \hspace{1cm} (15)

and

\[ \text{Cov}(\bar{y}_{\text{CSS}}, \bar{y}_{\text{CSS}}) = \theta_1 \left[ 1 + (n-1)\rho_y \right]^{1/2} \left[ 1 + (n-1)\rho_x \right]^{1/2} \frac{S_{yx}}{nm} = \frac{\tilde{s}_{yx}}{m} \]  \hspace{1cm} (16)

The bias expressions of estimators \( \bar{y}_R, \bar{y}_p \) and \( \bar{y}_{lr} \) are given as

\[ \text{Bias}(\bar{y}_R) = \bar{y} \left[ \frac{1}{2} \tilde{C}_x^2 - \tilde{C}_{yx} \right] \]  \hspace{1cm} (17)

\[ \text{Bias}(\bar{y}_p) = \tilde{Y} \tilde{C}_{yx} \]  \hspace{1cm} (18)

\[ \text{Bias}(\bar{y}_{lr}) = 0 \]  \hspace{1cm} (19)

and the MSE expressions of estimators \( \bar{y}_R, \bar{y}_p \) and \( \bar{y}_{lr} \) are given as

\[ \text{MSE}(\bar{y}_R) = \bar{Y}^2 \left( \tilde{C}_y^2 + \tilde{C}_x^2 - 2\tilde{C}_{yx} \right) \]  \hspace{1cm} (20)

\[ \text{MSE}(\bar{y}_p) = \bar{Y}^2 \left( \tilde{C}_y^2 + \tilde{C}_x^2 + 2\tilde{C}_{yx} \right) \]  \hspace{1cm} (21)

\[ \text{MSE}(\bar{y}_{lr}) = \bar{Y}^2 \tilde{C}_y^2 \left( 1 - \tilde{\rho}_{yx}^2 \right) \]  \hspace{1cm} (22)

where, \( \tilde{\rho}_{yx}^2 = \frac{\tilde{s}_{yx}^2}{\tilde{s}_y^2 \tilde{s}_x^2} \), such that

\[ \tilde{s}_{yx} = \left( \frac{m-1}{m} \right) \tilde{s}_{yx} \text{ and } \tilde{s}_x^2 = \left( \frac{m-1}{m} \right) \tilde{s}_x^2 \]

with

\[ \tilde{C}_x^2 = \frac{\tilde{s}_x}{\bar{Y}^2} \text{ and } \tilde{C}_{yx} = \frac{\tilde{s}_{yx}}{\bar{Y} \bar{X}} \]  \hspace{1cm} (23)
4. Proposed Estimator

Motivated by Koyuncu (2012), we propose the following estimator for estimating population mean \( \bar{Y} \) of a study variable \( Y \) under CSS assuming \( \bar{X} \) is known as

\[
t_p = \left[ w_1 \bar{Y}_{\text{CSS}} + w_2 \left( \frac{\bar{X}_{\text{CSS}}}{\bar{X}} \right)^\gamma \right] \exp \left[ \frac{\eta (\bar{X} - \bar{X}_{\text{CSS}})}{\eta (\bar{X} + \bar{X}_{\text{CSS}}) + 2\lambda} \right]
\]  

(24)

Where \( \gamma \) is a suitable real number, \( \eta \) and \( \lambda \) are either real numbers or the functions of the known parameters associated with an auxiliary attribute. \((w_1, w_2)\) are suitably chosen scalars to be properly determined for minimum mean square error (MSE) of suggested estimators and \( w_1 + w_2 \neq 1 \). (See Sharma and Singh 2013a)

Expanding equation (24) in terms of \( e \)'s up to the first order of approximation, we have,

\[
t_p - \bar{Y} = \bar{Y}(w_1 - 1) + w_1 \bar{Y} e_0 + \left( w_2 + w_2 \bar{Y} e_1 + w_2 \frac{\gamma (\gamma - 1)}{2} e_1^2 \right) - \frac{1}{2} w_1 \bar{Y} \tau e_1 - \frac{1}{2} w_1 \bar{Y} \tau e_0 e_1 - \frac{1}{2} w_2 \tau e_1
\]

\[
- \frac{1}{2} w_2 \tau e_1^2 + \frac{3}{8} w_1 \bar{Y} \tau^2 e_1^2 + \frac{3}{8} w_2 \tau^2 e_1^2
\]

(25)

where, \( e_0 = \frac{\bar{Y}_{\text{CSS}}}{\bar{Y}} - 1 \), \( e_1 = \frac{\bar{X}_{\text{CSS}}}{\bar{X}} - 1 \) and \( \tau = \frac{\eta \bar{X}}{\eta \bar{X} + \lambda} \)

To obtain the bias and MSE of the estimator \( t_p \) to the first degree of approximation, we write

Such that, \( E(e_i) = 0 \); \( i = 0, 1 \).

also,

\[
E(e_0^2) = \tilde{C}_{y}^2, \quad E(e_1^2) = \tilde{C}_{x}^2, \quad \text{and} \quad E(e_0 e_1) = \tilde{C}_{yx},
\]

Taking expectation both sides of equation (25), we get the bias expression of estimator \( t_p \) as

\[
\text{Bias}(t_p) = \bar{Y} + w_1 \bar{Y} \left[ 1 - \frac{1}{2} \tau \tilde{C}_{yx} + \frac{3}{8} \tau^2 \tilde{C}_{x}^2 \right] + w_2 \left[ 1 + \left( \frac{1}{2} \gamma (\gamma - 1) - \frac{1}{2} \gamma \tau + \frac{3}{8} \tau^2 \right) \tilde{C}_{x}^2 \right]
\]

(26)

Squaring both sides of equation (25) and taking expectation we get the MSE expression of estimator \( t_p \) as
\[ \text{MSE}(t_p) = \bar{Y}^2 + \bar{Y}^2 w_1^2 A_1 + w_2^2 A_2 + 2w_1 \bar{Y} A_3 + 2w_2 \bar{Y} A_4 + 2w_1 w_2 \bar{Y} A_5 \]  

(27)

where,

\[
A_1 = 1 + \tilde{C}_y^2 + \tau^2 \tilde{C}_x^2 - 2\tau \tilde{C}_{yx}
\]

\[
A_2 = 1 + \left( \left( \gamma - \frac{1}{2} \tau \right)^2 + \gamma(\gamma - 1) - \gamma + \frac{3}{4} \tau^2 \right) \tilde{C}_x^2
\]

\[
A_3 = 1 - \frac{1}{2} \tilde{C}_{yx} + \frac{3}{8} \tau^2 \tilde{C}_x^2
\]

\[
A_4 = 1 + \left( \frac{1}{2} \gamma(\gamma - 1) - \frac{1}{2} \gamma \tau + \frac{3}{8} \tau^2 \right) \tilde{C}_x^2
\]

\[
A_5 = 1 + (\gamma - \tau) \tilde{C}_{yx} + \left( \frac{1}{2} \gamma(\gamma - 1) - \gamma \tau + \tau^2 \right) \tilde{C}_x^2
\]

Partially differentiating equation (27) with respect to \( w_1 \) and \( w_2 \) and equating to zero, we get the optimum value of \( w_1 \) and \( w_2 \) as

\[
w_1(\text{opt}) = \frac{A_4 A_5 - A_2 A_4}{A_1 A_2 - A_5^2}
\]

\[
w_2(\text{opt}) = \frac{A_3 A_5 - A_1 A_4}{A_1 A_2 - A_5^2}
\]

Now suppose, \( \bar{X} \) is unknown, the analogue of \( t_p \) becomes

\[
t_p = \left[ w_1 \bar{Y}_{\text{CSS}} + w_2 \left( \frac{x_{\text{CSS}}}{\bar{X}_{\text{CSS}}} \right)^{\gamma} \right] \exp \left[ \frac{\eta \left( x_{\text{CSS}} - \bar{X}_{\text{CSS}} \right)}{\eta \left( \bar{X}_{\text{CSS}} + x_{\text{CSS}} \right) + 2\lambda} \right]
\]

(28)

where, the notations used here are already defined earlier.

To obtain the bias and MSE of the proposed class of estimators \( T_1 \), we define

\[
e_1 = \frac{(x_{\text{CSS}} - \bar{X})}{\bar{X}}
\]

Such that
The expressions for bias and MSE of the proposed estimator $t'_p$ using CSS are given respectively as

$$\text{Bias}(t'_p) = \bar{Y} + w_1\bar{Y}\left[1 - \frac{1}{2}\tilde{C}_{yx} + \frac{3}{8}\tau^2\tilde{C}_x^2\right] + w_2\left[1 + \left(\gamma - 1\right) - \frac{1}{2}\tau + \frac{3}{8}\tau^2\right]\tilde{C}_x^2$$

$$\text{MSE}(t'_p) = \bar{Y}^2 \left[1 + \frac{1}{2}\tau + \frac{3}{8}\tau^2\right]\tilde{C}_x^2$$

where,

$$A_1 = 1 + \tilde{C}_y^2 + \tau^2\tilde{C}_x^2 - 2\tau\tilde{C}_{yx}$$

$$A_2 = 1 + \left(\gamma - \frac{1}{2}\tau\right)^2 + \frac{3}{4}\tau\tilde{C}_x^2$$

$$A_3 = 1 - \frac{1}{2}\tilde{C}_{yx} + \frac{3}{8}\tau^2\tilde{C}_x^2$$

$$A_4 = 1 + \left(\frac{1}{2}\gamma - 1\right) - \frac{1}{2}\gamma\tau + \frac{3}{8}\tau^2\tilde{C}_x^2$$

$$A_5 = 1 + \left(\gamma - \tau\right)\tilde{C}_{yx} + \left(\frac{1}{2}\gamma - 1\right) - \gamma\tau + \frac{3}{8}\tau^2\tilde{C}_x^2$$

Partially differentiating equation (31) with respect to $w_1$ and $w_2$ and equating to zero, we get the optimum value of $w_1$ and $w_2$ as

$$w_1(\text{opt}) = \frac{A_4A_5 - A_2A_3}{A_1A_2 - A_5^2}$$

$$w_2(\text{opt}) = \frac{A_4A_5 - A_1A_4}{A_1A_2 - A_5^2}$$

\textbf{Note:} It can be observed from equation (26), (27) and (30), (31) that the bias and MSE of $t_p$ and $t'_p$ look similar. However, due to single and double sampling design the dissimilarity exists only in terms \(\left(\tilde{C}_x^2, \tilde{C}_{yx}\right)\) and \(\left(\tilde{C}_x^2, \tilde{C}_{yx}\right)\).
Table 2 Members of class of estimators $t_p$

<table>
<thead>
<tr>
<th>Estimators</th>
<th>Constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{p1} = [w_1 Y_{CSS} + w_2] \exp \left( \frac{X - \bar{x}<em>{CSS}}{(X + \bar{x}</em>{CSS}) + 2} \right)$</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>$t_{p2} = [w_1 Y_{CSS} + w_2 \left( \frac{X}{\bar{x}<em>{CSS}} \right)] \exp \left[ \frac{X - \bar{x}</em>{CSS}}{(X + \bar{x}_{CSS}) + 2} \right]$</td>
<td></td>
</tr>
<tr>
<td>$t_{p3} = [w_1 Y_{CSS} + w_2 \left( \frac{X}{\bar{x}<em>{CSS}} \right)] \exp \left[ \frac{X - \bar{x}</em>{CSS}}{(X + \bar{x}_{CSS}) + 2S_X} \right]$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$t_{p4} = [w_1 Y_{CSS} + w_2 \left( \frac{X}{\bar{x}<em>{CSS}} \right)] \exp \left[ \frac{X - \bar{x}</em>{CSS}}{(X + \bar{x}_{CSS}) + 2\rho_X} \right]$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$t_{p5} = [w_1 Y_{CSS} + w_2 \left( \frac{X}{\bar{x}<em>{CSS}} \right)] \exp \left[ \frac{X - \bar{x}</em>{CSS}}{(X + \bar{x}_{CSS}) + 2C_X} \right]$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$t_{p6} = [w_1 Y_{CSS} + w_2] \exp \left( \frac{X - \bar{x}<em>{CSS}}{(X + \bar{x}</em>{CSS}) + 2C_X} \right)$</td>
<td></td>
</tr>
<tr>
<td>$t_{p7} = [w_1 Y_{CSS} + w_2] \exp \left[ \frac{X - \bar{x}<em>{CSS}}{(X + \bar{x}</em>{CSS}) + 2S_X} \right]$</td>
<td></td>
</tr>
<tr>
<td>$t_{p8} = [w_1 Y_{CSS} + w_2 \left( \frac{X}{\bar{x}<em>{CSS}} \right)] \exp \left[ \frac{\rho_X (X - \bar{x}</em>{CSS})}{\rho_X (X + \bar{x}_{CSS}) + 2S_X} \right]$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

5. **Empirical Study**

In order to check the efficiency of proposed estimators, we take a data set which is earlier considered by Koyuncu and Kadilar (2009) and Singh and Solanki (2013). The data concerns primary and secondary schools of 923 districts of Turkey in 2007. The description of variables is given below.
y = number of teachers in both primary and secondary school;
x = number of students in both primary and secondary school.

\[ N = 923 \quad n = 360 \quad n = 180 \quad m = 2 \quad \bar{X} = 11440.5 \quad \bar{Y} = 436.43 \]

\[ S_y = 749.94 \quad S_x = 21331.13 \quad \rho_{yx} = 0.9543 \quad \rho_y = -0.00255 \quad \rho_x = -0.00316 \]

For two-phases, one can select \(1 < m < 5\) (as we mentioned earlier \(1 < m < k\)). All possible values of \(m\) are considered. Here in this problem we have taken \(m = 2\). Following table shows the variance/MSE and PRE of all the estimators considered here.

where, \(\text{PRE}(\bar{y}) = \frac{V(\bar{y})}{\text{MSE}(\bar{y})} \times 100\).

**Table 3** Variance/Minimum MSE/PRE's of considered estimators under single and two phase sampling (using usual notation in two-phase as \(\bar{y}\) is \(\bar{y}'\) and so on)

<table>
<thead>
<tr>
<th>Estimators</th>
<th>CSS (Single phase)</th>
<th>CSS (Two phase)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>V/MSE</td>
<td>PRE</td>
</tr>
<tr>
<td>(\bar{y})</td>
<td>1696.4819</td>
<td>100</td>
</tr>
<tr>
<td>(\bar{y}_P)</td>
<td>6433.3297</td>
<td>26.3702</td>
</tr>
<tr>
<td>(\bar{y}_R)</td>
<td>151.9342</td>
<td>1116.6033</td>
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<tr>
<td>(\bar{y}_{1r})</td>
<td>151.5154</td>
<td>1119.6765</td>
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<tr>
<td>(t_{p1})</td>
<td>35.2478</td>
<td>4813.0127</td>
</tr>
<tr>
<td>(t_{p2})</td>
<td>81.7762</td>
<td>2074.5416</td>
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<tr>
<td>(t_{p3})</td>
<td>48.8614</td>
<td>3472.0529</td>
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<tr>
<td>(t_{p4})</td>
<td>81.7811</td>
<td>2074.4179</td>
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<tr>
<td>(t_{p5})</td>
<td>81.7720</td>
<td>2074.6482</td>
</tr>
<tr>
<td>(t_{p6})</td>
<td>35.2426</td>
<td>4813.7321</td>
</tr>
<tr>
<td>(t_{p7})</td>
<td>33.7716</td>
<td>5023.4031</td>
</tr>
</tbody>
</table>

\[ t_{p7} = 4.3375 \quad 39111.6850 \quad 13.2356 \quad 12817.6050 \]
6. Conclusion

In this paper, we proposed an efficient class of exponential estimators $t^p$ using auxiliary variable in circular systematic sampling (CSS). The Variance/MSE/minimum MSE and PRE of different estimators have been shown in Table 3 and it has been observed that the estimator $t^p$, which is a member of proposed family of estimators $t^p$ has minimum MSE among all the estimators considered here. Thus, from empirical study, we conclude that estimator $t^p$ is more efficient than other existing estimators in CSS for single-phase and two-phase sampling for the given data set.

References

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