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Research Article

Nonparametric Estimation of Hazard Function with Functional Explicatory Variable in Single Functional Index

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Abstract

We introduce a nonparametric estimate of the conditional hazard function, when the covariate is functional. We prove consistency properties (with rates) in various situations, including censored and independent variables. The rates of convergence emphasize the crucial role played by the small ball probabilities with respect to the distribution of the explanatory functional variable.

Keywords: Censored data; Conditional Hazard Function; Functional Variable; Nonparametric Estimation; Single Functional Index Process; Small Ball Probability

2000 Mathematics Subject Classification: Primary: 62G05, Secondary: 62G07, 62G20, 62N02

1. Introduction

Estimation of hazard rate is an important issue in statistics, this topic should be approached under several angles depending on the complexity of the problem: possible presence of censorship in the observed sample (for instance a common phenomenon in medical applications), possible presence of dependency between the observed variables (for instance a common phenomenon in applications seismic or econometric) or presence of explanatory variable. Many techniques have been studied in the literature to deal with these different situations, but all of them deal only with real explanatory random variables or multi-dimensional.

Technical progress in collection and storage of data allow to have increasingly functional statistics: curves, images, tables, ... These data are modeled as realizations of a random variable taking its values in an abstract infinite dimensional space. In recent years the scientific community is naturally interested in the development of statistical tools able to handle this type of sample.

The single-index models are becoming increasingly popular, and have been paid considerable attention recently because of their importance in several areas of science such as econometrics, biostatistics, medicine, financial

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econometric and so on.

Thus, the estimation of a hazard rate in the presence of functional explanatory variables when the observations are linked with a single-index structure is a topical issue in which this paper proposes to make an initial response.

The layout of the paper is as follows: After a brief literature presented in Section 2.1, conditional Hazard in case of explanatory functional is introduced in section , models of the hazard rate for functional single-index in the case of non-censored-censored data are presented in sections 2.3 and 2.4. Estimators that we define are based on the techniques of convolution kernel.

In this paper, we propose to study the asymptotic behavior of these models; the pointwise almost complete convergence (with the rate) in the case of non-censored data (respectively censored data) 3.1 (resp. 3.2) and the uniform almost complete convergence (with the rate) in the case of non-censored data (respectively censored data) 4.1 (resp. 4.2)

In non-censored case, properties of the estimator of the conditional hazard function are obtained relatively easily from the known literature in estimating distribution function and conditional density literature. Thus, the proof of the results of section 3 will be presented synthetically using up the existing literature. contrariwise, the most interesting part of censored variables, these asymptotic properties are not obtained directly and to improve the readability of sections 3.2 and 4.2, technical details of the proofs which contains are shown at the end of paper.

2. Setting the Problem

2.1 *Bibliographic context*

If X is a random variable associated to a lifetime (ie, a random variable with values in $\mathbb{R}^+$, the hazard rate of X (sometimes called hazard function, failure or survival rate) is defined at point x as the instantaneous probability that life ends at time x. Specifically, we have:

$$h(x) = \lim_{\Delta x \longrightarrow 0} \frac{P\left(X \leq x + \Delta x | X \geq x\right)}{\Delta x}. \quad (x > 0) \tag{1}$$

When X has a density f with respect to the measure of Lebesgue, it is easy to see that the hazard rate can be written, as follows:

$$h(x) = \frac{f(x)}{S(x)}, \text{ for all } x \text{ such that } F(x) < 1, \tag{2}$$

where F denotes the distribution function of X and $S = 1 - F$ the survival function of X.
In many practical situations, we may have an explanatory variable Z and the main issue is to estimate the conditional random rate defined as

$$h^Z(x) = \lim_{\Delta x \longrightarrow 0} \frac{P\left(X \leq x + \Delta x | X > x,\ Z\right)}{\Delta x}, \text{ for } x > 0$$

which can be written naturally as follows:

$$h^Z(x) = \frac{f^Z(x)}{S^Z(x)}, \text{once } F^Z(x) < 1. \tag{3}$$

Study of functions h and h^Z is of obvious interest in many fields of science (biology, medicine, reliability , seismology, econometrics, ...) and many authors are interested in construction of nonparametric estimators of h . One of the most common techniques for building estimators of h (respectively h^Z) is based on (2) (resp. (3))

and consist in studying a quotient between the estimator of f (respectively f^Z) and that of S (respectively, S^Z). Patil *et al.* [17] presented an overview of these estimation techniques. Nonparametric methods based on the ideas of the convolution kernel, which are known for their good behavior in density estimation (conditional or not) problems are widely used in nonparametric estimation of hazard function. A wide range of literature in this area is provided by bibliographic reviews Singpurwalla and Wong [21] Hassani *et al.* [10], Izenman [11], Gefeller and Michels [9], Pascu and Vaduva [18], and Ferraty *et al.*.

2.2 *Conditional hazard in the case of explanatory functional*

The progress of data collection methods offers opportunities for statisticians to provide increasingly observations of functional variables. Works of Ramsay and Silverman [20] and Ferraty and Vieu [8] offer a wide range of statistics methods, parametric or nonparametric, recently developed to treat various estimation problems which occur in functional random variables (ie with values in a space of infinite dimension). Until now such statistical developments for functional variables in single functional index does not exist in the context of estimating a hazard function.

Let $(X_i, Z_i)_{1\leq i\leq n}$ be n random variables, identically distributed as the random pair (X, Z) with values in $\mathbb{R}\times\mathcal{H}$, where $\mathcal{H}$ is a separable real Hilbert space with the norm $\|\cdot\|$ generated by an inner product $<\cdot,\cdot>$. We consider the semi metric d_θ, associated to the single index $\theta\in\mathcal{H}$ defined by $\forall z_1, z_2\in\mathcal{H}$: $d_\theta(z_1, z_2) := |<z_1 - z_2, \theta>|$. Under such topological structure and for a fixed functional θ, we suppose that the conditional hazard function of X given $Z = z$ denoted by $h^z(\cdot)$ exists and is given by

$$\forall x\in\mathbb{R},\quad h^z_\theta(x) =: h(x| <z,\theta>).$$

Clearly, the identifiability of the model is assured, and we have for all $z\in\mathcal{H}$,

$$h_1(\cdot| <z,\theta_1>) = h_2(\cdot| <z,\theta_2>) \implies h_1 \equiv h_2 \quad \text{and} \quad \theta_1 = \theta_2.$$

For more details see Aït Saidi *et al.* [1]. In the following, we denote by $h(\theta,\cdot,Z)$, the conditional hazard function of X given $<z,\theta>$.

The objective of this paper is to study a model in which the conditional random explanatory variable Z is not necessarily real or multi-dimensional but only assumed to be values in an abstract space $\mathcal{H}$ provided a scalar product $<.,\cdot>$. As with any problem of non-parametric estimation, the dimension of the space $\mathcal{H}$ plays an important role in the properties of concentration of the variable X. Thus, when the dimension is not necessarily finite, probability functions defined by small balls of:

$$\phi_{\theta,z}(h) = \mathbb{P}\left(Z\in B_\theta(z,h)\right) = \mathbb{P}\left(Z\in\{z'\in\mathcal{H}, 0<|<z-z',\theta>|) < h\}\right),$$

intervene directly in the asymptotic behavior of any functional non-parametric estimator (see Ferraty and Vieu [8]). The asymptotic results presented later in this article on the estimation of the function $h(\theta,x,Z)$ does not escape this rule.

From now, z denotes a fixed element of the functional space $\mathcal{H}$, $\mathcal{N}_z$ denotes a fixed neighborhood of z and $\mathcal{S}_\mathbb{R}$ is a fixed compact of $\mathbb{R}^+$. Now, we should make some assumptions on the concentration function $\phi_{\theta,z}$:

(H1) $\forall h>0,\ \phi_{\theta,z}(h)>0.$

The non-parametric model on the estimated function h^Z will be determined by the regularity conditions on the conditional distribution of X knowing Z. These conditions are the following:

(H2) $\exists A_{\theta,z}<\infty, \exists b_1, b_2>0, \forall(x_1,x_2)\in\mathcal{S}^2_\mathbb{R},\ \forall(z_1,z_2)\in\mathcal{N}^2_z$:

$$|F(\theta,x_1,z_1) - F(\theta,x_2,z_2)| \leq A_{\theta,z}\left(\|z_1,z_2\|^{b_1} + |x_1-x_2|^{b_2}\right),$$
$$|f(\theta,x_1,z_1) - f(\theta,x_2,z_2)| \leq A_{\theta,z}\left(\|z_1,z_2\|^{b_1} + |x_1-x_2|^{b_2}\right);$$

(H3) $\exists \nu < \infty, \forall (x, z') \in \mathcal{S}_{\mathbb{R}} \times \mathcal{N}_z,\ \ f(\theta, x, z') \leq \nu;$

(H4) $\exists \beta > 0,\ \forall (x, z') \in \mathcal{S}_{\mathbb{R}} \times \mathcal{N}_z,\ \ F(\theta, x, z') \leq 1 - \beta.$

2.3 *Construction of the estimator in the case of non-censured data*

Let $(X_i, Z_i)_{1 \leq i \leq n}$ be random variables, each of them follows the same law of a couple (X, Z) where X is valued in $\mathbb{R}$ and Z has values in the Hilbert space $(\mathcal{H}, < \cdot; \cdot >)$. In this section we will suppose that X_i and Z_i are observed.

Recent advances in non-parametric statistics for functional variables, as presented in Ferraty and Vieu [8] show that the techniques based on convolution kernels are easily transposed to the context of functional variables. Moreover, these kernel's techniques have good properties in the problems of estimation of hazard function when the variables are of finite-dimensional. The reader may consult the work Ferraty *et al.* [7] which is a pioneering paper on the subject and that of Quintela-del-Rio [19] for the most recent results in this area.

Therefore, drawing on these ideas, it is natural to try to construct an estimator of the function $h(\theta, \cdot, Z)$. To estimate the conditional distribution function and the conditional density in the presence of functional the variable Z, Mahiddine *et al.* [14] proposed the following functional kernel estimators:

$$\widehat{F}(\theta, x, z) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1}(< z - Z_i, \theta >)\right) H\left(h_H^{-1}(x - X_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< z - Z_i, \theta >)\right)},$$

and

$$\widehat{f}(\theta, x, z) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1}(< z - Z_i, \theta >)\right) H'\left(h_H^{-1}(x - X_i)\right)}{h_H \sum_{i=1}^{n} K\left(h_K^{-1}(< z - Z_i, \theta >)\right)}.$$

where K is a kernel, H is a distribution function and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) is a sequence of positive real numbers.a kernel estimator of the functional conditional hazard function $h(\theta, \cdot, Z)$ may therefore be constructed in the following way:

$$\widehat{h}(\theta, x, Z) = \frac{\widehat{f}(\theta, x, Z)}{1 - \widehat{F}(\theta, x, Z)}. \tag{4}$$

The assumptions we need later for the parameters of the estimator, ie on K, H, h_H and h_K are not restrictive. Indeed, on one hand, they are not specific to the problem of estimating $h(\theta, x, Z)$ (but rather inherent to the estimation problems of $F(\theta, x, Z)$ and $f(\theta, x, Z)$), and in other hand they correspond to the assumptions usually made in the context of non-functional variables. More precisely, we introduce the following conditions which guarantee the good behavior of the estimators $\widehat{F}(\theta, x, Z)$ and $\widehat{f}(\theta, x, Z)$ (see Ferraty and Vieu [8]):

(H5) H is a bounded Lipschitz continuous function, such that

$$\int H'(t)dt = 1, \quad \int |t|^{b_2} H(t)dt < \infty, \quad \text{and} \int H^2(t)dt < \infty$$

(H6) K is positive bounded function with support $[-1, 1]$.

(H7) The bandwidth h_K has to satisfy

$$\lim_{n\to\infty} h_K = 0 \text{ and } \lim_{n\to\infty} \frac{\log n}{n h_H \phi_{\theta,x}(h_K)} = 0,$$

(H8) The bandwidth h_H has to satisfy

$$\lim_{n\to\infty} h_H = 0 \text{ et } \exists a > 0,\ \lim_{n\to\infty} n^a\, h_H = \infty.$$

Under these general conditions, we will establish in 3.1 the pointwise convergence of the kernel estimator $\widehat{h}(\theta, x, z)$ of the functional conditional hazard function $h(\theta, x, z)$ when the observed sample is not censored. In section 3.2, these results will be generalized to censored variables.

2.4 *Estimation in censored case*

Estimation of the hazard function when the data are censored is an important problem in medical research. So, in practice, in medical applications, it can be in the presence of variables censored. This problem is usually modeled by considering a positive variable called C, and the observed random variables are not the couples (X_i, Z_i) but only the (T_i, Δ_i, Z_i) where $T_i = \text{minimize}(X_i,\ C_i)$ and $\Delta_i = I_{X_i \le C_i}$. In the following we use the notations $F_1(\theta, \cdot, Z)$ and $f_1(\theta, \cdot, Z)$ to describe the distribution function and conditional density of C knowing Z and we use the notation $S_1(\theta, \cdot, Z) = 1 - F_1(\theta, \cdot, Z)$. Models such censorship where abundantly studied in the literature for real or multi-dimensional random variables, and in the nonparametric case kernel's techniques are particularly used (see Tanner and Wong [22] Padgett [16] Lecoutre and Ould-Said [13] and van Keilegom Veraverbeke [23]), for functional variables see Ferraty *et al.*, and Laksaci and Mechab [12] in the case of spatial variables.

The aim of this section, is to adapt these ideas as part of an explanatory variable Z functional, and build a kernel estimator function type of conditional random $h(\theta, \cdot, Z)$ adapted to the censored data. If we introduce the notation $L(\theta, \cdot, Z) = 1 - S_1(\theta, \cdot, Z) S(\theta, \cdot, Z)$ and $\varphi(\theta, \cdot, Z) = f(\theta, \cdot, Z) S_1(\theta, \cdot, Z)$, we can reformulate the expression (3) as follows:

$$h(\theta, t, Z) = \frac{\varphi(\theta, t, Z)}{1 - L(\theta, t, Z)},\ \forall t,\ L(\theta, t, Z) < 1. \tag{5}$$

So, we can define function estimators $\varphi(\theta, \cdot, Z)$ and $L(\theta, \cdot, Z)$ by setting

$$\widehat{L}(\theta, t, Z) = \frac{\displaystyle\sum_{i=1}^{n} K\left(h_K^{-1}(< z - Z_i, \theta >)\right) H\left(h_H^{-1}(t - T_i)\right)}{\displaystyle\sum_{i=1}^{n} K\left(h_K^{-1}(< z - Z_i, \theta >)\right)}$$

and

$$\widehat{\varphi}(\theta, t, Z) = \frac{\displaystyle\sum_{i=1}^{n} K\left(h_K^{-1}(< z - Z_i, \theta >)\right) \Delta_i H'\left(h_H^{-1}(t - T_i)\right)}{\displaystyle h_H \sum_{i=1}^{n} K\left(h_K^{-1}(< z - Z_i, \theta >)\right)}.$$

Finally the hazard function estimator is given as:

$$\widetilde{h}(\theta, t, Z) = \frac{\widehat{\varphi}(\theta, t, Z)}{1 - \widehat{L}(\theta, t, Z)}. \tag{6}$$

In addition to the assumptions introduced in section 2.3, we need additional conditions. These assumptions are identical to those found in the classical literature for non-functional variables (see previous references), these additional hypotheses are as follows:

(H9) Conditionally to Z, the variables X and C are independent;

(H10) $\exists A_{\theta,z} < \infty, \exists b_1, b_2 > 0, \forall (t_1, t_2) \in \mathcal{S}_{\mathbb{R}}^2,\ \forall (z_1, z_2) \in \mathcal{N}_z^2$:

$$|L(\theta, t_1, z_1) - L(\theta, t_2, z_2)| \le A_{\theta,z} \left(\|z_1 - z_2\|^{b_1} + |t_1 - t_2|^{b_2} \right)$$
$$|\varphi(\theta, t_1, z_1) - \varphi(\theta, t_2, z_2)| \le A_{\theta,z} \left(\|z_1 - z_2\|^{b_1} + |t_1 - t_2|^{b_2} \right);$$

(H11) $\exists \mu < \infty,\ \varphi(\theta, t, z') < \mu,\ \forall (t, z') \in \mathbb{R}_+ \times \mathcal{N}_z$,

(H12) $\exists \eta > 0,\ L(\theta, t, z') \le 1 - \eta,\ \ \forall (t, z') \in \mathbb{R}_+ \times \mathcal{N}_z$.

Under these very general conditions, we establish in Section 3.1 the rates of convergence of the kernel estimator $\widetilde{h}(\theta, \cdot, z)$ of the functional conditional Hazard function $h(\theta, \cdot, z)$ when couples of variables $(X_i, Z_i)_{i=1,\dots,n}$ are independents. In section 3.1 these results will be generalized by dispensing with the condition of censored data.

3. Pointwise Almost Complete Convergence

3.1 *Case of non censored data*

We begin by studying statistical samples satisfying a classical assumption of independence, couples (X_i, Z_i) are iid

Theorem 3.1. Under hypotheses (H1)-(H8), we have:

$$\sup_{x \in \mathcal{S}_{\mathbb{R}}} |\widehat{h}(\theta, x, z) - h(\theta, x, z)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2} \right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\, h_H\, \phi_{\theta,z}(h_K)}} \right),$$

Proof. The proof is based on the following decomposition, valid for any $x \in \mathcal{S}_{\mathbb{R}}$:

$$\begin{aligned}
\widehat{h}(\theta, x, z) - h(\theta, x, z) &= \frac{1}{(1 - \widehat{F}(\theta, x, z))(1 - F(\theta, x, z))} \left(\widehat{f}(\theta, x, z) - f(\theta, x, z) \right) \\
&\quad + \frac{f(\theta, x, z)}{(1 - \widehat{F}(\theta, x, z))(1 - F(\theta, x, z))} \left(\widehat{F}(\theta, x, z) - F(\theta, x, z) \right) \\
&\quad - \frac{F(\theta, x, z)}{(1 - \widehat{F}(\theta, x, z))(1 - F(\theta, x, z))} \left(\widehat{f}(\theta, x, z) - f(\theta, x, z) \right), \\
&= \frac{1}{1 - \widehat{F}(\theta, x, z)} \left(\widehat{f}(\theta, x, z) - f(\theta, x, z) \right) \\
&\quad + \frac{h(\theta, x, z)}{1 - \widehat{F}(\theta, x, z)} \left(\widehat{F}(\theta, x, z) - F(\theta, x, z) \right)
\end{aligned}$$

hence

$$\begin{aligned}
\sup_{x \in \mathcal{S}_{\mathbb{R}}} \left| \widehat{h}(\theta, x, z) - h(\theta, x, z) \right| &\le \frac{1}{\inf_{x \in \mathcal{S}_{\mathbb{R}}} \left| 1 - \widehat{F}(\theta, x, z) \right|} \left(\sup_{x \in \mathcal{S}_{\mathbb{R}}} \left| \widehat{f}(\theta, x, z) - f(\theta, x, z) \right| \right) \\
&\quad + \frac{\sup_{x \in \mathcal{S}_{\mathbb{R}}} |h(\theta, x, z)|}{\inf_{x \in \mathcal{S}_{\mathbb{R}}} \left| 1 - \widehat{F}(\theta, x, z) \right|} \left(\sup_{x \in \mathcal{S}_{\mathbb{R}}} \left| \widehat{F}(\theta, x, z) - F(\theta, x, z) \right| \right).
\end{aligned}$$

which leads to a constant $C < \infty$:

$$\sup_{x\in\mathcal{S}_\mathbb{R}} \left|\widehat{h}(\theta,x,z) - h(\theta,x,z)\right| \leq \quad C\,\frac{\left\{\sup_{x\in\mathcal{S}_\mathbb{R}}\left(\left|\widehat{f}(\theta,x,z) - f(\theta,x,z)\right| + \left|\widehat{F}(\theta,x,z) - F(\theta,x,z)\right|\right)\right\}}{\inf_{x\in\mathcal{S}_\mathbb{R}}\left|1-\widehat{F}(\theta,x,z)\right|}.$$

And conventionally (see for instance the Proposition A6ii of Ferraty and Vieu [8]) the announced result follows directly from the following properties:

$$\sup_{x\in\mathcal{S}_\mathbb{R}} |F(\theta,x,z) - \widehat{F}(\theta,x,z)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co}\left(\sqrt{\frac{\log n}{n\,\phi_{\theta,z}(h_K)}}\right), \tag{7}$$

and

$$\sup_{x\in\mathcal{S}_\mathbb{R}} |f(\theta,x,z) - \widehat{f}(\theta,x,z)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,h_H\,\phi_{\theta,z}(h_K)}}\right), \tag{8}$$

and from the next result which is a consequence of property (7).

Corollary 3.2. Under the conditions of Theorem 3.1, we have

$$\exists\delta > 0 \text{ such that } \sum_{n=1}^{\infty}\mathbb{P}\left\{\inf_{x\in\mathcal{S}_\mathbb{R}}\left|1-\widehat{F}(\theta,x,z)\right| < \delta\right\} < \infty.$$

The results (7) and (8) are known results (see for instance Ferraty and Vieu [8], Propositions 6.19 and 6.20).

3.2 *Estimation with censored data*

The goal now is to take these asymptotic properties in the broader context of a censored sample as described in Section 2.4. We will begin in this section by discussing the case censored. Obviously, obtaining these results require more sophisticated than those presented under uncensored technical developments. To ensure a good readability in this Section 3.2, the presentation of these technical details will later in Paragraph 5.
We begin by studying statistical samples satisfying a standard assumption of independence, ie. triples (X_i, C_i, Z_i) are i.i.d.

Theorem 3.3. Under assumptions (H1)-(H2), and (H5)-(H12), we have:

$$\sup_{t\in\mathcal{S}_\mathbb{R}} |\widetilde{h}(\theta,t,z) - h(\theta,t,z)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,h_H\,\phi_{\theta,z}(h_K)}}\right),$$

Proof. The result is based on the bellow decomposition , wherein $C$is a real constant strictly positive:

$$\begin{aligned}
\sup_{t\in\mathcal{S}_\mathbb{R}}\left|\widetilde{h}(\theta,t,z) - h(\theta,t,z)\right| \leq \quad & \frac{1}{\inf\limits_{t\in\mathcal{S}_\mathbb{R}}\left|1-\widehat{L}(\theta,t,z)\right|}\left\{\sup_{t\in\mathcal{S}_\mathbb{R}}|\widehat{\varphi}(\theta,t,z) - \varphi(\theta,t,z)|\right\} \\
& + \frac{\sup\limits_{t\in\mathcal{S}_\mathbb{R}}|h(\theta,t,z)|}{\inf\limits_{t\in\mathcal{S}_\mathbb{R}}\left|1-\widehat{L}(\theta,t,z)\right|}\left\{\sup_{t\in\mathcal{S}_\mathbb{R}}\left|\widehat{L}(\theta,t,z) - L(\theta,t,z)\right|\right\}. \\
\leq \quad & C\,\frac{\sup\limits_{t\in\mathcal{S}_\mathbb{R}}\left\{|\widehat{\varphi}(\theta,t,z) - \varphi(\theta,t,z)| + \left|L(\theta,t,z) - \widehat{L}(\theta,t,z)\right|\right\}}{\inf\limits_{t\in\mathcal{S}_\mathbb{R}}\left|1-\widehat{L}(\theta,t,z)\right|},
\end{aligned} \tag{9}$$

which is obtained from (3) and (5) proceeding as to establish (17). Since $\widehat{L}(\theta,t,Z)$ is none other than the kernel estimator of the conditional distribution function of T knowing Z is obtained directly (see Ferraty and Vieu [8], Proposition 6.19) that:

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{L}(\theta,t,Z)-L(\theta,t,Z)\right|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_{\theta,z}(h_K)}}\right). \tag{10}$$

The proprieties of the estimator $\widehat{\varphi}(\theta,\cdot,Z)$ are given in Lemma 3.4, the desired result is obtained directly from (9)-(12).

Lemma 3.4. Under hypotheses of Theorem 3.3, we have:

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}}\left|\widehat{\varphi}(\theta,t,Z)-\varphi(\theta,t,Z)\right|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,h_H\,\phi_{\theta,z}(h_K)}}\right). \tag{11}$$

The next result which is a consequence of property (10).

Corollary 3.5. Under the conditions of Theorem 3.3, we have

$$\exists\delta>0 \text{ such that } \sum_{n=1}^{\infty}\mathbb{P}\left\{\inf_{x\in\mathcal{S}_{\mathbb{R}}}\left|1-\widehat{L}(\theta,x,z)\right|<\delta\right\}<\infty. \tag{12}$$

4. Uniform Almost Complete Convergence

In this party we derive the uniform version of Theorem 3.1. The study of the uniform consistency is motivated by the fact that the latter is an indispensable tool for studying the asymptotic properties of all estimates of the functional index if is unknown. Noting that, in the multivariate case, the uniform consistency is a standard extension of the pointwise one, however, in our functional case, it requires some additional tools and topological conditions (see Ferraty *et al.* [6], for more discussion on the uniform convergence in nonparametric functional statistics). Thus, in addition to the conditions introduced previously, we need the following ones. Firstly, consider

$$\mathcal{S}_{\mathcal{H}}\subset\bigcup_{k=1}^{d_n^{\mathcal{S}_{\mathcal{H}}}}B(x_k,r_n) \text{ and } \Theta_{\mathcal{H}}\subset\bigcup_{j=1}^{d_n^{\Theta_{\mathcal{H}}}}B(t_j,r_n)$$

with x_k (resp. t_j) $\in\mathcal{H}$ and $r_n, d_n^{\mathcal{S}_{\mathcal{H}}}, d_n^{\Theta_{\mathcal{H}}}$ are sequences of positive real numbers which tend to infinity as n goes to infinity.

4.1 *Non censored data*

Thereafter we propose to study the uniform almost complete convergence of our estimator defined above (4) for this, we need the following assumptions:

(A1) There exists a differentiable function $\phi(\cdot)$ such that $\forall z\in\mathcal{S}_{\mathcal{H}}$ and for all $\theta\in\Theta_{\mathcal{H}}$,

$$0<C\phi(h)\le\phi_{\theta,z}(h)\le C'\phi(h)<\infty \text{ and } \exists\eta_0>0,\ \forall\eta<\eta_0,\ \phi'(\eta)<C,$$

(A2) $\forall(x_1,x_2)\in\mathcal{S}_{\mathbb{R}}\times\mathcal{S}_{\mathbb{R}},\forall(z_1,z_2)\in\mathcal{S}_{\mathcal{H}}\times\mathcal{S}_{\mathcal{H}}$ and $\forall\theta\in\Theta_{\mathcal{H}}$,

$$|F(\theta,x_1,z_1)-F(\theta,x_2,z_2)|\le A\left(\|z_1,z_2\|^{b_1}+|x_1-x_2|^{b_2}\right),$$
$$|f(\theta,x_1,z_1)-f(\theta,x_2,z_2)|\le A\left(\|z_1,z_2\|^{b_1}+|x_1-x_2|^{b_2}\right);$$

(A3) $\exists \nu < \infty, \forall (x,z') \in \mathcal{S}_{\mathbb{R}} \times \mathcal{N}_z,\ \ \forall \theta \in \Theta_{\mathcal{H}},\ \ f(\theta,x,z') \leq \nu;$

(A4) $\exists \beta > 0,\ \forall (x,z') \in \mathcal{S}_{\mathbb{R}} \times \mathcal{N}_z,\ \ \forall \theta \in \Theta_{\mathcal{H}},\ \ F(\theta,x,z') \leq 1-\beta.$

(A5) The kernel K satisfy (H3) and Lipschitz's condition holds

$$|K(u) - K(v)| \leq C\|u - v\|,$$

(A6) For $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$ the sequences $d_n^{\mathcal{S}_{\mathcal{H}}}$ and $d_n^{\Theta_{\mathcal{H}}}$ satisfy:

$$\frac{(\log n)^2}{n\phi(h_K)} < \log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}} < \frac{n\phi(h_K)}{\log n},$$

and $\sum_{n=1}^{\infty} n^{1/2b_2} (d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})^{1-\beta} < \infty$ for some $\beta > 1$

(A7) For some $\gamma \in (0,1)$, $\lim_{n\to\infty} n^{\gamma} h_H = \infty$, and for $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$ the sequences $d_n^{\mathcal{S}_{\mathcal{F}}}$ and $d_n^{\Theta_{\mathcal{F}}}$ satisfy:

$$\frac{(\log\ n)^2}{nh_H\phi(h_K)} < \log\ d_n^{\mathcal{S}_{\mathcal{F}}} + \log d_n^{\Theta_{\mathcal{F}}} < \frac{nh_H\phi(h_K)}{\log n},$$

and $\sum_{n=1}^{\infty} n^{(3\gamma+1)/2} (d_n^{\mathcal{S}_{\mathcal{F}}} d_n^{\Theta_{\mathcal{F}}})^{1-\beta} < \infty$, for some $\beta > 1$

Remark 4.1. Note that Assumptions (A1)-(A4) are, respectively, the uniform version of (H1)-(H4). Assumptions (A1) and (A6) are linked with the the topological structure of the functional variable, see Ferraty *et al.* [5].

Theorem 4.2. Under hypotheses (A1)-(A7) and (H5), we have:

$$\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{z\in\mathcal{S}_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathbb{R}}} |\widehat{h}(\theta,x,z) - h(\theta,x,z)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{nh_H\phi(h_K)}}\right)$$

In the particular case, where the functional single-index is fixed we get the following result.

Corollary 4.3. Under Assumptions (A1)-(A7) and (H4), as n goes to infinity, we have

$$\sup_{z\in\mathcal{S}_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathbb{R}}} |\widehat{h}(\theta,x,z) - h(\theta,x,z)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}}{nh_H\phi(h_K)}}\right)$$

Proof of theorem 4.2. Clearly The proofs of these two results namely the Theorem 4.2 and Corollary 4.3 can be deduced from the following intermediate results which are only uniform version of properties (7) and (8).

$$\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{z\in\mathcal{S}_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathbb{R}}} |\widehat{F}(\theta,x,z) - F(\theta,x,z)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right), \qquad (13)$$

and

$$\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{z\in\mathcal{S}_{\mathcal{H}}} \sup_{x\in\mathcal{S}_{\mathbb{R}}} |\widehat{f}(\theta,x,z) - f(\theta,x,z)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{nh_H\phi(h_K)}}\right) \qquad (14)$$

and from the next result which is a consequence of property (13).

Corollary 4.4. Under the conditions of Theorem 4.2, we have

$$\exists \delta > 0 \text{ such that } \sum_{n=1}^{\infty} \mathbb{P}\left\{\inf_{z\in\mathcal{S}_{\mathcal{H}}} \inf_{x\in\mathcal{S}_{\mathbb{R}}} |1 - \widehat{F}(\theta,x,z)| < \delta\right\} < \infty.$$

The results (13) and (14) are known results (see for example Mahiddine *et al.* [14]).

4.2 *Censored data*

Thereafter we propose to study the uniform almost complete convergence of our estimator defined above (6) for this, we need the following assumptions:

(A2a) $\forall(t_1,t_2)\in\mathcal{S}_{\mathbb{R}}\times\mathcal{S}_{\mathbb{R}},\forall(z_1,z_2)\in\mathcal{S}_{\mathcal{H}}\times\mathcal{S}_{\mathcal{H}}$ and $\forall\theta\in\Theta_{\mathcal{H}}$,

$$|L(\theta,t_1,z_1)-L(\theta,t_2,z_2)|\leq A\left(\|z_1,z_2\|^{b_1}+|t_1-t_2|^{b_2}\right),$$
$$|\varphi(\theta,t_1,z_1)-\varphi(\theta,t_2,z_2)|\leq A\left(\|z_1,z_2\|^{b_1}+|t_1-t_2|^{b_2}\right);$$

(A3a) $\exists\nu<\infty,\forall(t,z')\in\mathcal{S}_{\mathbb{R}}\times\mathcal{N}_z,\ \ \forall\theta\in\Theta_{\mathcal{H}},\ \ \varphi(\theta,t,z')\leq\nu;$

(A4a) $\exists\beta>0,\ \forall(t,z')\in\mathcal{S}_{\mathbb{R}}\times\mathcal{N}_z,\ \ \forall\theta\in\Theta_{\mathcal{H}},\ \ L(\theta,t,z')\leq 1-\beta.$

Theorem 4.5*.* Under hypotheses (A1), (A5)-(A7) and (A2a)-(A4a), we get:

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{z\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widetilde{h}(\theta,t,z)-h(\theta,t,z)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{nh_H\phi(h_K)}}\right)$$

In the particular case, where the functional single-index is fixed we get the following result.

Corollary 4.6*.* Under Assumptions (A1), (A5)-(A7), (A2a)-(A4a) and (H4), as n goes to infinity, we have

$$\sup_{z\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widetilde{h}(\theta,t,z)-h(\theta,t,z)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}}{nh_H\phi(h_K)}}\right)$$

Poof of theorem 4.5. The result is based on the decomposition (9). Clearly The proofs of these two results namely the Theorem 4.5 and Corollary 4.6 can be deduced from the following intermediate results which are only uniform version of properties (10) and (11).
The properties of the estimators $\widehat{L}(\theta,\cdot,z)$ and $\widehat{\varphi}(\theta,\cdot,z)$ are given in the following Lemma 4.7.
Finally, the desired result is obtained directly from (9), (15), (16).

Lemma 4.7*.* Under hypotheses of Theorem 4.5, we have:

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{z\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widehat{L}(\theta,t,z)-L(\theta,t,z)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right),\tag{15}$$

and

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{z\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widehat{\varphi}(\theta,t,z)-\varphi(\theta,t,z)|=\mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{nh_H\phi(h_K)}}\right)\tag{16}$$

The next result which is a consequence of property (15).

Corollary 4.8*.* Under the conditions of Theorem 4.5, we have

$$\exists\delta>0 \text{ such that } \sum_{n=1}^{\infty}\mathbb{P}\left\{\inf_{z\in\mathcal{S}_{\mathcal{H}}}\inf_{t\in\mathcal{S}_{\mathbb{R}}}|1-\widehat{L}(\theta,t,z)|<\delta\right\}<\infty.$$

Sketch of Proof of Lemma 4.7

- The proof of (15) is based on some results depending on the following decomposition;

$$\begin{aligned}\widehat{L}(\theta,t,z)-L(\theta,t,z) &= \frac{1}{\widehat{\varphi}_D(\theta,z)}\left\{\left(\widehat{L}_N(\theta,t,z)-\mathbb{E}\widehat{L}_N(\theta,t,z)\right)-\left(L(\theta,t,z)-\mathbb{E}\widehat{L}_N(\theta,t,z)\right)\right\}\\ &\quad+\frac{L(\theta,t,z)}{\widehat{\varphi}_D(\theta,z)}\left\{1-\widehat{\varphi}_D(\theta,z)\right\} \qquad (17)\end{aligned}$$

Then the rest of the proof is similar the one given in Mahiddine *et al.*, where, it is sufficient to replace $\widehat{F}_D(\theta,z)$, $F(\theta,t,z)$ and $\mathbb{E}(\widehat{F}_N(\theta,t,z))$ (Lemma 6, corollary 3 and Lemma 7) by $\widehat{\varphi}_D(\theta,z)$,, $L(\theta,t,z)$ and $\mathbb{E}(\widehat{L}_N(\theta,t,z))$ respectively.

Then the rest is deduced directly from Lemma 4.10, Lemma 4.11 and Corollary 4.9.

Corollary 4.9. Under Assumptions (A1), (A5) and (A6), we have as $n\to\infty$

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{z\in\mathcal{S}_{\mathcal{H}}}|\widehat{\varphi}_D(\theta,z)-1|=\mathcal{O}_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right), \qquad (18)$$

and

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{z\in\mathcal{S}_{\mathcal{H}}}\widehat{\varphi}_D(\theta,z)<\frac{1}{2}\right)<\infty \qquad (19)$$

Lemma 4.10. Under Assumptions (A1), (A2) and (H5), we have, as n goes to infinity

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{z\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|L(\theta,t,z)-\mathbb{E}(\widehat{L}_N(\theta,t,z))|=\mathcal{O}(h_K^{b_1}+h_H^{b_2}) \qquad (20)$$

Lemma 4.11. Under assumptions (A1), (A5)-(A7) and (A2a)-(A4a) we have, as n goes to infinity

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{z\in\mathcal{S}_{\mathcal{H}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widehat{L}_N(\theta,t,z)-\mathbb{E}\left[\widehat{L}_N(\theta,t,z)\right]|=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}+log\, d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \qquad (21)$$

- $\rightsquigarrow$ Concerning (16) the proof is based at first on the following decomposition;

$$\begin{aligned}\widehat{\varphi}(\theta,t,z)-\varphi(\theta,t,z) &= \frac{1}{\widehat{\varphi}_D(\theta,z)}\left(\widehat{\varphi}_N(\theta,t,z)-\mathbb{E}(\widehat{\varphi}_N(\theta,t,z))\right)\\ &\quad-\frac{1}{\widehat{\varphi}_D(\theta,z)}\left(\varphi(\theta,t,z)-\mathbb{E}\widehat{\varphi}_N(\theta,t,z)\right)\\ &\quad+\frac{\varphi(\theta,t,z)}{\widehat{\varphi}_D(\theta,z)}\left(1-\widehat{\varphi}_D(\theta,z)\right)\end{aligned}$$

The rest is deduced directly from Lemma 4.12, Lemma 4.13 and Corollary 4.9.

Lemma 4.12. Under Assumptions (A1), (A2a) and (H5), we have, as n goes to infinity

$$\sup_{\theta\in\Theta_{\mathcal{F}}}\sup_{z\in\mathcal{S}_{\mathcal{F}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\varphi(\theta,t,z)-\mathbb{E}(\widehat{\varphi}_N(\theta,t,z))|=\mathcal{O}(h_K^{b_1}+h_H^{b_2}) \qquad (22)$$

Lemma 4.13. Under the assumptions (A1), (A5) ,(A2a), (A7) and (H5), we have, as n goes to infinity

$$\sup_{\theta\in\Theta_{\mathcal{F}}}\sup_{z\in\mathcal{S}_{\mathcal{F}}}\sup_{t\in\mathcal{S}_{\mathbb{R}}}|\widehat{\varphi}_N(\theta,t,z)]-\mathbb{E}\left[\widehat{\varphi}_N(\theta,t,z)\right]|=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{F}}}+\log d_n^{\Theta_{\mathcal{F}}}}{nh_H\phi_{\theta,z}(h_K)}}\right) \qquad (23)$$

5. Proofs of Technical Lemmas

In what follows C and c denote generic strictly positive real constants. Furthermore, the following notation are introduced:

$$K_i(\theta,z) = K(h_K^{-1}(< z - Z_i, \theta >)),\ H_i'(t) = H'\left(h_H^{-1}(t - T_i)\right),$$

$$\widehat{\varphi}_N(\theta,t,z) = \frac{1}{nh_H \mathbb{E}K_1(\theta,z)} \sum_{i=1}^{n} K_i(\theta,z) H_i'(t)\Delta_i,$$

$$\widehat{\varphi}_D(\theta,z) = \frac{1}{n\mathbb{E}K_1(z)} \sum_{i=1}^{n} K_i(\theta,z),$$

$$V_i = \frac{1}{\mathbb{E}K_1(\theta,z)} K_i(\theta,z),$$

$$W_i = \frac{1}{h_H \mathbb{E}K_1(\theta,z)} K_i(\theta,z) H_i'(t)\Delta_i,$$

Proof of Corollary 3.2. It is clear that

$$\begin{aligned}\inf_{x\in\mathcal{S}_{\mathbb{R}}} |1 - \widehat{F}(\theta,x,z)| \quad &\leq \quad \left(1 - \sup_{x\in\mathcal{S}_{\mathbb{R}}} F(\theta,x,z)\right) \Big/ 2 \\ &\Rightarrow \quad \sup_{x\in\mathcal{S}_{\mathbb{R}}} |\widehat{F}(\theta,x,z) - F(\theta,x,z)| \geq \left(1 - \sup_{x\in\mathcal{S}_{\mathbb{R}}} F(\theta,x,z)\right) \Big/ 2.\end{aligned}$$

which implies that

$$\begin{aligned}&\sum_{n=1} \mathbb{P}\left\{\inf_{x\in\mathcal{S}_{\mathbb{R}}} |1 - \widehat{F}(\theta,x,z)| \leq \left(1 - \sup_{x\in\mathcal{S}_{\mathbb{R}}} F(\theta,x,z)\right) \Big/ 2\right\} \\ &\leq \sum_{n=1} \mathbb{P}\left\{\sup_{x\in\mathcal{S}_{\mathbb{R}}} |\widehat{F}(\theta,x,z) - F(\theta,x,z)| \geq \left(1 - \sup_{x\in\mathcal{S}_{\mathbb{R}}} F(\theta,x,z)\right) \Big/ 2\right\} < \infty.\end{aligned}$$

We deduce from property (7) that

$$\sum_{n=1} \mathbb{P}\left\{\inf_{x\in\mathcal{S}_{\mathbb{R}}} |1 - \widehat{F}(\theta,x,z)| \leq \left(1 - \sup_{x\in\mathcal{S}_{\mathbb{R}}} F(\theta,x,z)\right) \Big/ 2\right\} < \infty.$$

This proof is achieved by taking $\delta = (1 - \sup_{x\in\mathcal{S}_{\mathbb{R}}} F(\theta,x,z))/2$ which is strictly positive.
Proof of Lemma 3.4. By using the following decomposition:

$$\widehat{\varphi}(\theta,t,z) - \varphi(\theta,t,z) = \frac{\left(\widehat{\varphi}_N(\theta,t,z) - \varphi_N(\theta,t,z)\right)\varphi_D(\theta,z) - \left(\widehat{\varphi}_D(\theta,z) - \varphi_D(\theta,z)\right)\varphi_N(\theta,t,z)}{\widehat{\varphi}_D(\theta,z)\varphi_D(\theta,z)},$$

and under the Proposition A6ii de Ferraty and Vieu [8], the result of Lemma 3.4 will result directly following three properties:

$$|\widehat{\varphi}_D(\theta,z) - 1| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\, h_H\, \phi_{\theta,z}(h_K)}}\right), \tag{24}$$

$$\sup_{t\in\mathcal{S}_{\mathbb{R}}} |\mathbb{E}\widehat{\varphi}_N(\theta,t,z) - \varphi(\theta,t,z)| = \mathcal{O}(h_K^{b_1} + h_H^{b_2}), \tag{25}$$

and

$$\frac{1}{\widehat{\varphi}_D(z)} sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi}_N(\theta,t,z) - \mathbb{E}\widehat{\varphi}_N(\theta,t,z)| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\, h_H\, \phi_{\theta,z}(h_K)}}\right), \tag{26}$$

- **Proof of (24).** It suffices to note that we can write

$$\widehat{\varphi}_D(\theta, z) = \frac{1}{n}\sum_{i=1}^{n} V_i,$$

with

$$|V_i| \quad = \quad \mathcal{O}\left(\frac{1}{\phi_{\theta,z}(h)}\right), \tag{27}$$

and

$$\mathbb{E}V_i^2 \quad = \quad \mathcal{O}\left(\frac{1}{\phi_{\theta,z}(h)}\right). \tag{28}$$

By applying an exponential inequality for bounded variables (for example Corollary A9i of Ferraty and Vieu [8]) and taking into account the results (27) et (28), we arrive at

$$\mathbb{P}\left[|\widehat{\varphi}_D(\theta, z) - \mathbb{E}\widehat{\varphi}_D(\theta, z)| > \varepsilon\sqrt{\frac{\log n}{n\,\phi_{\theta,z}(h_K)}}\right] = \mathcal{O}(n^{-C\varepsilon^2}).$$

Now simply choose ε large enough to get the result (24).

- **Proof of (25).** We have, for any $t \in \mathcal{S}_{\mathbb{R}}$:

$$\begin{aligned}
\mathbb{E}\widehat{\varphi}_N(\theta, t, z) &= \frac{1}{h_H \mathbb{E}K_1(\theta, z)}\mathbb{E}\left(K_1(\theta, z) H_1'(t)\Delta_1\right) \\
&= \frac{1}{h_H \mathbb{E}K_1(\theta, z)}\mathbb{E}\left(K_1(\theta, z)\mathbb{E}\left(H_1'(t) I_{X_1 \leq C_1}\big| < Z_1, \theta >\right)\right) \\
&= \frac{1}{h_H \mathbb{E}K_1(\theta, z)}\mathbb{E}\left(K_1(z) E(H_1(t) S_1(\theta, X_1, Z_1)| < Z_1, \theta >)\right), \qquad (29)
\end{aligned}$$

the last equality arising of conditional independence between C_1 and X_1 introduced into (H9), furthermore we have

$$\begin{aligned}
E(H_1(t) S_1(\theta, X_1, z)| < Z_1, \theta >) &= \int H'(\frac{t-u}{h_H}) S_1(\theta, u, Z_1) f(\theta, u, Z_1) du \\
&= h_H \int H'(v)\varphi(\theta, t - v h_H, Z_1) dv \\
&= h_H \left(\varphi(\theta, t, z) + o(h_H^{b_2} + h_K^{b_1})\right), \qquad (30)
\end{aligned}$$

the last equality resulting from the property of Lipschitz function φ^z introduced in (H10) and the fact that H' is a probability density. It should be noted, again because of the condition (H10), that them $o()$ involved in the result (30) are uniform for $t \in \mathcal{S}_{\mathbb{R}}$. Thus, the result (25) is an immediate consequence of (29) and (30).

- **Proof of (26).** The compactness of the set $\mathcal{S}_{\mathbb{R}}$ can be covered by the u_n disjoint intervals as follows:

$$\mathcal{S}_{\mathbb{R}} \subset \cup_{m=1}^{u_n}]\tau_m - l_n, \tau_m + l_n[,$$

where $\tau_1, \ldots, \tau_{u_n}$ are points of $\mathcal{S}_{\mathbb{R}}$ and where l_n and u_n are chosen such that

$$\exists C > 0, \exists c > 0, l_n = C u_n^{-1} = n^{-\alpha}. \tag{31}$$

For each $t \in \mathcal{S}_{\mathbb{R}}$ noting τ_t the single τ_m such as $t \in [\tau_m - l_n, \tau_m + l_n[$. Finally, (26) can be easily deduced from the following results:

$$\frac{1}{\widehat{\varphi}_D(\theta,z)} \sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi}_N(\theta,t,z) - \widehat{\varphi}_N(\theta,\tau_t,z)| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,h_H\,\phi_{\theta,z}(h_K)}}\right), \tag{32}$$

$$\frac{1}{\widehat{\varphi}_D(\theta,z)} \sup_{t\in\mathcal{S}_{\mathbb{R}}} |\mathbb{E}\widehat{\varphi}_N(\theta,t,z) - \mathbb{E}\widehat{\varphi}_N(\theta,\tau_t,z)| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,h_H\,\phi_{\theta,z}(h_K)}}\right), \tag{33}$$

and

$$\frac{1}{\widehat{\varphi}_D(\theta,z)} \sup_{t\in\mathcal{S}_{\mathbb{R}}} |\widehat{\varphi}_N(\theta,\tau_t,z) - \mathbb{E}\widehat{\varphi}_N(\theta,\tau_t,z)| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\,h_H\,\phi_{\theta,z}(h_K)}}\right). \tag{34}$$

- **Proof of (32).** Because of the condition (H5),

  there is exist a finite constant C such that for all $t \in \mathcal{S}_{\mathbb{R}}$:

$$\begin{aligned} |\widehat{\varphi}_N(\theta,t,z) - \widehat{\varphi}_N(\theta,\tau_t,z)| &= \frac{1}{nh_H\mathbb{E}K_1(\theta,z)} \sum_{i=1}^{n} \Delta_i K_i(z)\,(H_i'(t) - H_i'(\tau_t)) \\ &\leq \frac{C}{nh_H\mathbb{E}K_1(\theta,z)} \sum_{i=1}^{n} K_i(\theta,z)\frac{|t-\tau_t|}{h_H} \\ &\leq C\widehat{\varphi}_D(\theta,z)l_n h_H^{-2}. \end{aligned} \tag{35}$$

  By using (31) and choosing c large enough, we obtain directly (32).

- **Proof of (33).** This result is obtained directly from (24) and (35) using Proposition A6ii of Ferraty and Vieu [8].

- **Proof of (34).** Obtaining (34) is based on the use of an exponential inequality. Specifically, it suffices to note that we can write

$$\widehat{\varphi}_N(\theta,t,z) = \frac{1}{n}\sum_{i=1}^{n} W_i,$$

  with

$$|W_i| = \mathcal{O}\left(\frac{1}{h_H\phi_{\theta,z}(h)}\right), \tag{36}$$

  and

$$\begin{aligned} \mathbb{E}W_i^2 &= \frac{1}{h_H^2(\mathbb{E}K_1(\theta,z))^2}\mathbb{E}K_i^2(\theta,z)H_i^{'2}(t)\Delta_i^2 \\ &\leq C\frac{1}{h_H^2(\mathbb{E}K_1(\theta,z))^2}\mathbb{E}\left(K_i^2(\theta,z)\mathbb{E}(H_i^{'2}(t))\big| < Z_i,\theta >\right) \\ &\leq C\frac{1}{h_H\phi_{\theta,z}(h)^2}\mathbb{E}\left(K_i^2(\theta,z)\int\frac{1}{h_H}H'\left(\frac{t-u}{h_H}\right)^2 f(\theta,u,Z_i)du\right) \\ &= \mathcal{O}\left(\frac{1}{h_H\phi_{\theta,z}(h)}\right). \end{aligned} \tag{37}$$

By using the condition (31) we arrive at

$$\mathbb{P}\left[\sup_{x\in\mathcal{S}_{\mathbb{R}}}|\widehat{\varphi}_N(\theta,\tau_t,z)-\mathbb{E}\widehat{\varphi}_N(\theta,\tau_t,z)|>\varepsilon\sqrt{\frac{\log n}{n\,h_H\,\phi_{\theta,z}(h_K)}}\right]\leq$$
$$n^{\alpha}\max_{m=1,\dots u_n}\mathbb{P}\left[|\widehat{\varphi}_N(\theta,\tau_m,z)-\mathbb{E}\widehat{\varphi}_N(\theta,\tau_m,z)|>\varepsilon\sqrt{\frac{\log n}{n\,h_H\,\phi_{\theta,z}(h_K)}}\right]. \quad (38)$$

Moreover, by applying an exponential inequality to bounded variables (for example the corollary A9i by Ferraty and Vieu [8]) and taking into account the results (36) and (37), we arrive at

$$\mathbb{P}\left[|\widehat{\varphi}_N(\theta,\tau_m,z)-\mathbb{E}\widehat{\varphi}_N(\theta,\tau_m,z)|>\varepsilon\sqrt{\frac{\log n}{n\,h_H\,\phi_\theta z(h_K)}}\right]=\mathcal{O}(n^{-C\varepsilon^2}). \quad (39)$$

It suffices now to choose ϵ large enough to directly obtain the desired result from (38) and from (39).

The results (32), (33) and (34) are sufficient to conclude the proof of the result (26).
Finally, Lemma 3.4 is a consequence of (24), (25) and (26) and decomposition (5).
Proof of Corollary 4.9.

- Concerninig (18) For all $z\in\mathcal{S}_{\mathcal{H}}$ and $\theta\in\Theta_{\mathcal{H}}$, we set

$$k(z)=\arg\text{minimize}_{k\in\{1\dots r_n\}}\|z-z_k\| \text{ and } j(\theta)=\arg\text{minimize}_{j\in\{1\dots l_n\}}\|\theta-t_j\|.$$

Let us consider the following decomposition

$$\begin{aligned}\sup_{\theta\in\mathcal{S}_{\mathcal{H}}}\sup_{\Theta\in\mathcal{S}_{\mathcal{H}}}|\widehat{\varphi}_D(\theta,z)-\mathbb{E}\left(\widehat{\varphi}_D(\theta,z)\right)| \leq & \underbrace{\sup_{\theta\in\mathcal{S}_{\mathcal{H}}}\sup_{\Theta\in\mathcal{S}_{\mathcal{H}}}\left|\widehat{\varphi}_D(\theta,z)-(\widehat{\varphi}_D(\theta,z_{k(z)})\right|}_{\Pi_1}\\ &+\underbrace{\sup_{\theta\in\mathcal{S}_{\mathcal{H}}}\sup_{\Theta\in\mathcal{S}_{\mathcal{H}}}\left|\widehat{\varphi}_D(\theta,z_{k(z)})-\widehat{\varphi}_D(t_{j(\theta)},z_{k(z)})\right|}_{\Pi_2}\\ &+\underbrace{\sup_{\theta\in\mathcal{S}_{\mathcal{H}}}\sup_{\Theta\in\mathcal{S}_{\mathcal{H}}}\left|\widehat{\varphi}_D(t_{j(\theta)},z_{k(z)})-\mathbb{E}\left(\widehat{\varphi}_D(t_{j(\theta)},z_{k(z)})\right)\right|}_{\Pi_3}\\ &+\underbrace{\sup_{\theta\in\mathcal{S}_{\mathcal{H}}}\sup_{\Theta\in\mathcal{S}_{\mathcal{H}}}\left|\mathbb{E}\left(\widehat{\varphi}_D(t_{j(\theta)},z_{k(z)})\right)-\mathbb{E}\left(\widehat{\varphi}_D(\theta,z_{k(z)})\right)\right|}_{\Pi_4}\\ &+\underbrace{\sup_{\theta\in\mathcal{S}_{\mathcal{H}}}\sup_{\Theta\in\mathcal{S}_{\mathcal{H}}}\left|\mathbb{E}\left(\widehat{\varphi}_D(\theta,z_{k(z)})\right)-\mathbb{E}\left(\widehat{\varphi}_D(\theta,z)\right)\right|}_{\Pi_5}\end{aligned}$$

For Π_1 and Π_2, we employe the Hölder continuity condition on K, Cauchy Schwartz's and the Bernstein's inequalities, we get

$$\Pi_1=\mathcal{O}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right),\ \Pi_2=\mathcal{O}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}+\log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \quad (40)$$

Then, by using the fact that $\Pi_4 \leq \Pi_1$ and $\Pi_5 \leq \Pi_2$, we get for n tending to infinity

$$\Pi_4 = \mathcal{O}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right), \ \Pi_5 = \mathcal{O}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right) \tag{41}$$

Now, we deal with Π_3, for all $\eta > 0$, we have

$$\mathbb{P}\left(\Pi_3 > \eta\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)\right)$$

$$\leq d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}} \max_{k\in\{1\ldots d_n^{S_{\mathcal{H}}}\}} \max_{j\in\{1\ldots d_n^{\Theta_{\mathcal{H}}}\}} \mathbb{P}\left(\left|\widehat{\varphi}_D(t_{j(\theta)}, z_{k(z)}) - \mathbb{E}\left(\widehat{\varphi}_D(t_{j(\theta)}, z_{k(z)})\right)\right| > \eta\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)\right).$$

Applying Bernstein's exponential inequality to

$$\frac{1}{\phi(h_K)}\left(K_i(t_{j(\theta)}, z_{k(z)}) - \mathbb{E}\left(K_i(t_{j(\theta)}, z_{k(z)})\right)\right),$$

then under (A7), we get

$$\Pi_3 = \mathcal{O}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right).$$

Lastly the result will be easily deduced from the latter together with (40) and (41).

- Concerninig (19) It is easy to see that,

$\inf_{\theta\in\Theta_{\mathcal{H}}} \inf_{z\in\mathcal{S}_{\mathcal{H}}} |\widehat{\varphi}_D(\theta, z)| \leq 1/2 \implies \exists z \in \mathcal{S}_{\mathcal{H}},\ \exists\theta \in \Theta_{\mathcal{H}}$, such that

$$1 - \widehat{\varphi}_D(\theta, z) \geq 1/2 \implies \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{z\in\mathcal{S}_{\mathcal{H}}} |1 - \widehat{\varphi}_D(\theta, z)| \geq 1/2.$$

We deduce from (18) the following inequality

$$\mathbb{P}\left(\inf_{\theta\in\Theta_{\mathcal{H}}} \inf_{z\in\mathcal{S}_{\mathcal{H}}} |\widehat{\varphi}_D(\theta, z)| \leq 1/2\right) \leq \mathbb{P}\left(\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{z\in\mathcal{S}_{\mathcal{H}}} |1 - \widehat{\varphi}_D(\theta, z)| \leq 1/2\right).$$

Consequently,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\inf_{\theta\in\Theta_{\mathcal{H}}} \inf_{z\in\mathcal{S}_{\mathcal{H}}} \widehat{\varphi}_D(\theta, z) < \frac{1}{2}\right) < \infty$$

Proof of Lemma 4.7.

- Concerning (20), one has

$$\begin{aligned}\mathbb{E}\widehat{L}_N(\theta, t, z) - L(\theta, t, z) &= \frac{1}{\mathbb{E}K_1(z, \theta)}\mathbb{E}\left[\sum_{i=1}^{n} K_i(z, \theta) H_i(t)\right] - L(\theta, t, z) \\ &= \frac{1}{\mathbb{E}K_1(z, \theta)}\mathbb{E}\left(K_1(z, \theta)\left[E\left(H_1(t)| < Z_1, \theta >\right) - L(\theta, t, z)\right]\right). \end{aligned} \tag{42}$$

Moreover, we have

$$\mathbb{E}\left(H_1(t)| < Z_1,\theta >\right) = \int_{\mathbb{R}} H\left(h_H^{-1}(t-z)\right) f(\theta,z,Z_1)dz,$$

now, integrating by parts and using the fact that H is a *cdf*, we obtain

$$\mathbb{E}\left(H_1(t)| < Z_1,\theta >\right) \;=\; \int_{\mathbb{R}} H'(t)L(\theta,t-h_H t,Z_1)dt.$$

Thus, we have

$$\left|\mathbb{E}\left(H_1(t)| < Z_1,\theta >\right) - L(\theta,t,z)\right| \;\leq\; \int_{\mathbb{R}} H^{(1)}(t)\left|L(\theta,t-h_H t,Z_1) - L(\theta,t,z)\right| dt.$$

Finally, the use of (A2) implies that

$$\left|\mathbb{E}\left(H_1(t)| < Z_1,\theta >\right) - L(\theta,t,z)\right| \;\leq\; C\int_{\mathbb{R}} H'(t)\left(h_K^{b_1} + |t|^{b_2}h_H^{b_2}\right) dt. \tag{43}$$

Because this inequality is uniform on $(\theta,t,z) \in \Theta_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathbb{R}}$ and because of (H5), (20) is a direct consequence of (42), (43) and (19).

- Concerning (21), we keep the notation of the Corollary 4.9 and we use the compact of $\mathcal{S}_{\mathbb{R}}$, we can write that, for some, $t_1,\ldots,t_{u_n} \in \mathcal{S}_{\mathbb{R}}$, $\mathcal{S}_{\mathbb{R}} \subset \bigcup_{m=1}^{u_n}(t_m - l_n, t_m + l_n)$ with $l_n = n^{-\frac{1}{2b_2}}$ and $u_n \leq Cn^{\frac{1}{2b_2}}$. Taking $m(t) = \arg\text{minimize}_{\{1,2,\ldots,u_n\}}|t-t_m|$.

Thus, we have the following decomposition:

$$\begin{aligned}
\left|\widehat{L}_N(\theta,t,z) - \mathbb{E}\left(\widehat{L}_N(\theta,t,z)\right)\right| \leq\; & \underbrace{\left|\widehat{L}_N(\theta,t,z) - \widehat{L}_N(\theta,t,z_{k(z)})\right|}_{\Gamma_1} \\
& + \underbrace{\left|\widehat{L}_N(\theta,t,z_{k(z)}) - \mathbb{E}\left(\widehat{L}_N(\theta,t,z_{k(z)})\right)\right|}_{\Gamma_2} \\
& +2\underbrace{\left|\widehat{L}_N(t_{j(\theta)},t,z_{k(z)}) - \widehat{L}_N(t_{j(\theta)},t_{m(t)},z_{k(z)})\right|}_{\Gamma_3} \\
& +2\underbrace{\left|\mathbb{E}\left(\widehat{L}_N(t_{j(\theta)},t,z_{k(z)})\right) - \mathbb{E}\left(\widehat{L}_N(t_{j(\theta)},t_{m(t)},z_{k(z)})\right)\right|}_{\Gamma_4} \\
& + \underbrace{\left|\mathbb{E}\left(\widehat{L}_N(\theta,t,z_{k(z)})\right) - \mathbb{E}\left(\widehat{L}_N(\theta,t,z)\right)\right|}_{\Gamma_5}
\end{aligned}$$

$\hookrightarrow$ Concerning Γ_1 we have

$$\left|\widehat{L}_N(\theta,t,z) - \widehat{L}_N(\theta,t,z_{k(z)})\right| \leq \frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{\mathbb{E}K_1(\theta,z)}K_i(\theta,z)H_i(t) - \frac{1}{\mathbb{E}K_1(\theta,z_{k(z)})}K_i(\theta,z_{k(z)})H_i(t)\right|.$$

We use the Hölder continuity condition on K, the Cauchy-Schwartz inequality, the Bernstein's inequality and the boundness of H (assumption (H5)). This allows us to get:

$$\begin{aligned}\left|\widehat{L}_N(\theta,t,z)-\widehat{L}_N(\theta,t,z_{k(z)})\right| &\leq \frac{C}{\phi(h_K)}\frac{1}{n}\sum_{i=1}^{n}\left|K_i(\theta,z)H_i(t)-K_i(\theta,z_{k(z)})H_i(t)\right|\\ &\leq \frac{C}{\phi(h_K)}\frac{1}{n}\sum_{i=1}^{n}\left|H_i(t)\right|\left|K_i(\theta,z)-K_i(\theta,z_{k(z)})\right|\\ &\leq \frac{C'r_n}{\phi(h_K)}\end{aligned}$$

$\hookrightarrow$ Concerning Γ_2, the monotony of the functions $\mathbb{E}\widehat{L}_N(\theta,\cdot,z)$ and $\widehat{L}_N(\theta,\cdot,z)$ permits to write, $\forall m \leq u_n,\ \forall z\in\mathcal{S}_{\mathcal{H}},\ \forall\theta\in\Theta_{\mathcal{H}}$

$$\begin{aligned}\mathbb{E}\widehat{L}_N(\theta,t_{m(t)}-l_n,z_{k(z)}) \leq \sup_{t\in(t_{m(t)}-l_n,t_{m(t)}+l_n)}\mathbb{E}\widehat{L}_N(\theta,t,z)\leq \mathbb{E}\widehat{L}_N(\theta,t_{m(t)}+l_n,z_{k(z)})\\ \widehat{L}_N(\theta,t_{m(t)}-l_n,z_{k(z)}) \leq \sup_{t\in(t_{m(t)}-l_n,t_{m(t)}+l_n)}\widehat{L}_N(\theta,t,z)\leq \widehat{L}_N(\theta,t_{m(t)}+l_n,z_{k(z)}).\end{aligned}$$

Next, we use the Hölder's condition on $L(\theta,t,z)$ and we show that, for any $t_1,\ t_2\in\mathcal{S}_{\mathbb{R}}$ and for all $z\in\mathcal{S}_{\mathcal{H}},\ \theta\in\Theta_{\mathcal{H}}$

$$\begin{aligned}\left|\mathbb{E}\widehat{L}_N(\theta,t_1,z)-\mathbb{E}\widehat{L}_N(\theta,t_2,z)\right| &= \frac{1}{\mathbb{E}K_1(z,\theta)}\left|\mathbb{E}\left(K_1(z,\theta)L(\theta,t_1,Z_1)\right)-\mathbb{E}\left(K_1(z,\theta)L(\theta,t_2,Z_1)\right)\right|\\ &\leq C|t_1-t_2|^{b_2}.\end{aligned}$$

Now, we have, for all $\eta>0$

$$\mathbb{P}\left(\sup_{j\in\{1\ldots d_n^{\Theta_{\mathcal{H}}}\}}\sup_{k\in\{1\ldots d_n^{\mathcal{S}_{\mathcal{H}}}\}}\sup_{1\leq m\leq u_n}\left|\widehat{L}_N(\theta,t,z_{k(z)})-\mathbb{E}\widehat{L}_N(\theta,t,z_{k(z)})\right|>\eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)$$

$$=$$

$$\mathbb{P}\left(\max_{j\in\{1\ldots d_n^{\Theta_{\mathcal{H}}}\}}\max_{k\in\{1\ldots d_n^{\mathcal{S}_{\mathcal{H}}}\}}\max_{1\leq m\leq u_n}\left|\widehat{L}_N(\theta,t,z_{k(z)})-\mathbb{E}\widehat{L}_N(\theta,t,z_{k(z)})\right|>\eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)$$

$$\leq$$

$$u_nd_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}\max_{j\in\{1\ldots d_n^{\Theta_{\mathcal{H}}}\}}\max_{k\in\{1\ldots d_n^{\mathcal{S}_{\mathcal{H}}}\}}\max_{1\leq m\leq u_n}\mathbb{P}\left(\left|\widehat{L}_N(\theta,t,z_{k(z)})-\mathbb{E}\widehat{L}_N(\theta,t,z_{k(z)})\right|>\eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)$$

$$\leq$$

$$2u_nd_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}\exp\left(-C\eta^2\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}\right)$$

choosing $u_n=O\left(l_n^{-1}\right)=O\left(n^{\frac{1}{2b_2}}\right)$, we get

$$\mathbb{E}\left(\left|\widehat{L}_N(\theta,t,z_{k(z)})-\mathbb{E}\widehat{L}_N(\theta,t,z_{k(z)})\right|>\eta\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{n\phi(h_K)}}\right)\leq C'u_n\left(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}\right)^{1-C\eta^2}$$

putting $C\eta^2 = \beta$ and using (A4), we get

$$\Gamma_2 = \mathcal{O}_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{L}} d_n^{\Theta_\mathcal{L}}}{n\phi(h_K)}}\right).$$

$\hookrightarrow$ Concerning the terms Γ_3 and Γ_4, using Lipschitz's condition on the kernel H, one can write

$$\begin{aligned}\left|\widehat{L}_N(t_{j(\theta)}, t, z_{k(z)}) - \widehat{L}_N(t_{j(\theta)}, t_{m(t)}, z_{k(z)})\right| &\leq C\frac{1}{n\phi(h_K)}\sum_{i=1}^{n} K_i(t_{j(\theta)}, z_{k(z)})\left|H_i(t) - H_i(t_{m(t)})\right| \\ &\leq \frac{Cl_n}{nh_H\phi(h_K)}\sum_{i=1}^{n} K_i(t_{j(\theta)}, z_{k(z)}).\end{aligned}$$

Once again a standard exponential inequality for a sum of bounded variables allows us to write

$$\widehat{L}_N(t_{j(\theta)}, t, z_{k(z)}) - \widehat{L}_N(t_{j(\theta)}, t_{m(t)}, z_{k(z)}) = \mathcal{O}\left(\frac{l_n}{h_H}\right) + \mathcal{O}_{a.co}\left(\frac{l_n}{h_H}\sqrt{\frac{\log n}{n\phi_z(h_K)}}\right).$$

Now, the fact that $\lim_{n\to\infty} n^\gamma h_H = \infty$ and $l_n = n^{-1/2b_2}$ imply that:

$$\frac{l_n}{h_H\phi(h_K)} = o\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right),$$

then

$$\Gamma_3 = \mathcal{O}_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right).$$

Hence, for n large enough, we have

$$\Gamma_3 \leq \Gamma_4 = \mathcal{O}_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right).$$

$\hookrightarrow$ Concerning Γ_5, we have

$$\mathbb{E}\left(\widehat{L}_N(\theta, t, z_{k(z)})\right) - \mathbb{E}\left(\widehat{L}_N(\theta, t, z)\right) \leq \sup_{z\in\mathcal{S}_\mathcal{H}}\left|\widehat{L}_N(\theta, t, z) - \widehat{L}_N(\theta, t, z_{k(z)})\right|,$$

then following similar proof used in the study of Γ_1 and using the same idea as for $\mathbb{E}\left(\widehat{\varphi}_D(\theta, z_{k(z)})\right) - \mathbb{E}\left(\widehat{\varphi}_D(\theta, z)\right)$ we get, for n tending to infinity,

$$\Gamma_5 = \mathcal{O}_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}}}{n\phi(h_K)}}\right).$$

The proof of these for points are similar to ones given in Mahiddine at.al, so it is sufficient to replace $\widehat{F}_D(\theta, z)$, $F(\theta, t, z)$ and $\mathbb{E}(\widehat{F}_N(\theta, t, z))$ (Lemma 6, corollary 3 and Lemma 7) by $\widehat{\varphi}_D(\theta, z)$, $L(\theta, t, z)$ and $\mathbb{E}(\widehat{L}_N(\theta, t, z))$ respectively.

- Concerning(22), let $H_i^{'}(t) = H^{'}\left(h_H^{-1}(t - T_i)\right)$, note that

$$\mathbb{E}\widehat{\varphi}_N(\theta, t, z) - \varphi(\theta, t, z) = \frac{1}{h_H\,\mathbb{E}K_1(z,\theta)}\mathbb{E}\left(K_1(z,\theta)\left[\mathbb{E}\left(H_1'(t)I_{X_1 \le C_1}| < Z_1, \theta >\right) - h_H\varphi(\theta, t, z)\right]\right).$$

Moreover,

$$\begin{aligned}
\mathbb{E}\left(H_1'(t)S_1(\theta, X_1, z)| < Z_1, \theta >\right) &= \int_{\mathbb{R}} H'\left(h_H^{-1}(t - w)\right)S_1(\theta, w, Z_1)f(\theta, w, Z_1)dw,\\
&= h_H\int_{\mathbb{R}} H'\left(h_H^{-1}(t - w)\right)\varphi(\theta, w, Z_1)dw\\
&= h_H\int_{\mathbb{R}} H'(v)\varphi(\theta, t - vh_H, Z_1)dv.
\end{aligned}$$

Under condition (H10) we can write:

$$\left|\mathbb{E}\left(H_1'(t)S_1(\theta, X_1, z)| < Z_1, \theta >\right) - h_H\varphi(\theta, t, z)\right| \le h_H\int_{\mathbb{R}} H'(t)\left|\varphi(\theta, t - h_H t, Z_1) - \varphi(\theta, t, z)\right|dt.$$

Finally, (A2a) allows to write

$$\left|\mathbb{E}\left(H_1'(t)S_1(\theta, X_1, z)| < Z_1, \theta >\right) - h_H\varphi(\theta, t, z)\right| \le Ch_H\int_{\mathbb{R}} H'(t)\left(h_K^{b_1} + |t|^{b_2}h_H^{b_2}\right)dt.$$

This inequality is uniform on $(\theta, t, z) \in \Theta_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathbb{R}}$, now to finish the proof it is sufficient to use (H5).

- Concerning (23), let us keep the definition of $k(z)$ (resp. $j(\theta)$) as in Corollary 4.9. The compactness of $\mathcal{S}_{\mathbb{R}}$ permits to write that $\mathcal{S}_{\mathbb{R}} \subset \bigcup_{m=1}^{u_n}(t_m - l_n, t_m + l_n)$ with $l_n = n^{-\frac{3}{2}\gamma - \frac{1}{2}}$ and $u_n \le Cn^{\frac{3}{2}\gamma + \frac{1}{2}}$. Taking $m(t) = \text{arg minimize}_{\{1\ldots u_n\}}|t - t_m|$. Consider the following decomposition

$$\begin{aligned}
|\widehat{\varphi}_N(\theta, t, z) - \mathbb{E}\left(\widehat{\varphi}_N(\theta, t, z)\right)| = &\underbrace{\left|\widehat{\varphi}_N(\theta, t, z) - \widehat{\varphi}_N(\theta, t, z_{k(z)})\right|}_{\Delta_1}\\
&+\underbrace{\left|\widehat{\varphi}_N(\theta, t, z_{k(z)}) - \mathbb{E}\left(\widehat{\varphi}_N(\theta, t, z_{k(z)})\right)\right|}_{\Delta_2}\\
&+2\underbrace{\left|\widehat{\varphi}_N(t_{j(\theta)}, t, z_{k(z)}) - \widehat{\varphi}_N(t_{j(\theta)}, t_{j(t)}, z_{k(z)})\right|}_{\Delta_3}\\
&+2\underbrace{\left|\mathbb{E}\left(\widehat{\varphi}_N(t_{j(\theta)}, t, z_{k(z)})\right) - \mathbb{E}\left(\widehat{\varphi}_N(t_{j(\theta)}, t_{j(t)}, z_{k(z)})\right)\right|}_{\Delta_4}\\
&+\underbrace{\left|\mathbb{E}\left(\widehat{\varphi}_N(\theta, t, z_{k(z)})\right) - \mathbb{E}\left(\widehat{\varphi}_N(\theta, t, z)\right)\right|}_{\Delta_5}
\end{aligned}$$

$\rightsquigarrow$ Concerning Δ_1, we use the Hölder continuity condition on K, the Cauchy-Schwartz's inequality and the Bernstein's inequality. With theses arguments we get

$$\Delta_1 = \mathcal{O}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}}}{nh_H\phi(h_K)}}\right).$$

Then using the fact that $\Delta_5 \leq \Delta_1$, we obtain

$$\Delta_5 \leq \Delta_1 = \mathcal{O}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{nh_H\phi(h_K)}}\right). \tag{44}$$

$\rightsquigarrow$ For Δ_2, we follow the same idea given for Γ_2, we get

$$\Delta_2 = \mathcal{O}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{nh_H\phi(h_K)}}\right)$$

$\rightsquigarrow$ Concerning Δ_3 and Δ_4, using Lipschitz's condition on the kernel H,

$$\left|\widehat{\varphi}_N(t_{j(\theta)}, t, z_{k(z)}) - \widehat{\varphi}_N(t_{j(\theta)}, t_{m(t)}, z_{k(z)})\right| \leq \frac{l_n}{h_H^2\phi(h_k)},$$

using the fact that $\lim_{n\to\infty} n^\gamma h_H = \infty$ and choosing $l_n = n^{-\frac{3}{2}\gamma-\frac{1}{2}}$ implies

$$\frac{l_n}{h_H^2\phi(h_k)} = o\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{nh_H\phi(h_K)}}\right)$$

So, for n large enough, we have

$$\Delta_3 = \mathcal{O}_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{nh_H\phi(h_K)}}\right),$$

and as $\Delta_4 \leq \Delta_3$, we obtain

$$\Delta_4 \leq \Delta_3 = \mathcal{O}_{a.co}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}}}{nh_H\phi(h_K)}}\right). \tag{45}$$

Finally, the lemma can be easily deduced from (44) and (45).

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