Estimates for the Distribution of Semi-norms of $L_p(\Omega)$ Processes in Hölder Spaces

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Abstract

Zatula (2015) obtained estimates for the distribution of semi-norms of sample functions of $L_p(\Omega)$ processes, defined on a compact space, in Hölder spaces and studied the Hölder continuity of such processes. In the present article we provide estimates for distributions of semi-norms of sample functions of $L_p(\Omega)$ processes, defined on the infinite interval $[0, \infty)$, in Hölder spaces.

Keywords: $L_p(\Omega)$ processes; Moduli of continuity; Hölder spaces; Semi-norms

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1. Introduction

Let $(\mathbb{T}, \rho), \rho(t, s) = |t - s|, t, s \in \mathbb{T}$ be a metric space. Consider a random process $X = \{X(t), t \in \mathbb{T}\}$ and a function $f$ such that the following inequality holds

$$\limsup_{\varepsilon \downarrow 0} \frac{\sup_{0 < |t-s| \leq \varepsilon} |X(t) - X(s)|}{f(\varepsilon)} \leq 1$$

with probability 1.

The function $f(\varepsilon)$ is a modulus of continuity for the process $X$. A space of functions with moduli of continuity $f(\varepsilon)$ is the Hölder space and the functional

$$\sup_{0 < |t-s| \leq \varepsilon} \frac{|X(t) - X(s)|}{f(|t-s|)}$$

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is a semi-norm in the Hölder space. In the following we deal with estimates of distributions of semi-norms of sample functions of $L_p(\Omega)$ processes $X = \{X(t), \ t \in [0, \infty)\}$ in the Hölder spaces, i.e. probabilities

$$P\left( \sup_{0 < |t-s| \leq \epsilon \wedge t \leq \infty} \frac{|X(t) - X(s)|}{f(|t-s|)} > x \right).$$

Such estimates and assumptions under which semi-norms of sample functions of $L_p(\Omega)$ processes, defined on a compact space, satisfy the Hölder conditions were obtained by Zatula (2015). Similar results were provided for Gaussian processes, defined on a compact space, by Dudley (1973). Kozachenko (1985) generalized Dudley’s results for random processes belonging to Orlicz spaces, see also Buldygin and Kozachenko (2000), Zatula (2013). Marcus and Rosen (2008) obtained $L_p$ moduli of continuity for a wide class of continuous Gaussian processes. Kozachenko et al. (2011) studied the Lipschitz continuity of generalized sub-Gaussian processes and provided estimates for the distribution of Lipschitz norms of such processes. But all these problems were not considered yet for processes, defined on an infinite interval.

The Hölder continuity of random processes applies to the study of the rate of approximation of random functions by trigonometric polynomials. In particular, Kamenshchikova and Yanevich (2011) investigated an approximation of stochastic processes belonging to spaces $L_p(\Omega)$ by trigonometric sums in the space $L_q[0,2\pi]$.

The theory of sample path properties of non-stationary Gaussian processes based on concepts of the entropy and majorizing measures is now well studied. For an accessible introduction to these concepts and to the general theory of continuity, boundedness and suprema distributions for real-valued Gaussian processes, we refer to Adler (1990).

The method of majorizing measures is used in the theory of Gaussian stochastic processes for determining conditions of boundedness and sample path continuity with probability one of these processes. This method gives the possibility to obtain estimates for distributions of stochastic processes. Papers by Fernique (1971, 1975) are among the first ones in this direction. The method of majorizing measures turns out to be more effective to investigate various problems than the entropy method which was exploited by Dudley (1973). Talagrand (1987) found the necessary and sufficient conditions for sample path continuity or boundedness of Gaussian stochastic processes in terms of majorizing measures. Kozachenko and Moklyachuk (2003) used the method of majorizing measures for finding estimates of distribution of extremes of stochastic processes from the Orlicz space of random variables and estimates of the probability that a vector-valued stochastic process leaves a fixed region during a given interval of time.

A number of mathematicians studied various analytical properties of non-Gaussian processes and fields. Kozachenko and Moklyachuk (1999) determined and investigated special classes of square-Gaussian stochastic processes, i.e. processes which are formed by quadratic forms of Gaussian processes or by limits in the mean square of quadratic forms of Gaussian processes. Properties of random variables and processes belonging to the spaces $\text{Sub}_\varphi(\Omega)$ and $\text{SSub}_\varphi(\Omega)$ can be found in the book of Buldygin and Kozachenko (2000). Kozachenko et al. (2005) studied the upper estimate of
overrunning by \( \text{Sub}_p(\Omega) \) random process the level specified by a continuous function. The upper estimate of the probability that a random process belonging to the class \( V(\varphi, \psi) \) exceeds some function was obtained by Yamnenko and Vasylyk (2007). Conditions for the sample continuity of stochastic processes belonging to the spaces \( D_{V,W} \) were found by Kozachenko and Moklyachuk (2011).

Stochastic processes that take values in Orlicz spaces and properties of such processes were investigated by Rao and Ren (2002). Krinik and Swift (2004) studied different properties of exponential Orlicz spaces and Fenchel-Orlicz spaces. Estimates for the probability of a deviation in the uniform metric of sums of independent identically distributed fields, which belong to Orlicz spaces, were found by Kozachenko and Mlavets (2011). In all above-mentioned articles the authors consider spaces for which \( E|\xi|^p \) exists for each \( 1 \leq p < \infty \), but for \( L_p(\Omega) \) spaces similar problems haven’t been investigated, namely the task about estimates of distributions of semi-norms of sample functions of \( L_p(\Omega) \) processes in Hölder spaces.

2. Preliminaries

**Definition 2.1** A random variable \( \xi \) belongs to the space \( L_p(\Omega) \), \( 1 \leq p < \infty \), if the condition
\[
(E|\xi|^p)^{1/p} < \infty
\]
is satisfied.

It is well known that \( L_p(\Omega), 1 \leq p < \infty \) is a Banach space with the norm
\[
\|\xi\|_{L_p} = (E|\xi|^p)^{1/p}.
\]

**Definition 2.2** A random process \( X = \{X(t), t \in \mathbb{T}\} \) belongs to the space \( L_p(\Omega), 1 \leq p < \infty \), if for all \( t \in \mathbb{T} \) the random variable \( X(t) \in L_p(\Omega) \).

**Definition 2.3** ([3, 10]) A function \( q = \{q(t), t \in \mathbb{R}\} \) is called the modulus of continuity if \( q(t) \geq 0 \), \( q(0) = 0 \) and \( q(t+s) \leq q(t) + q(s) \) as \( t > 0, s > 0 \).

**Definition 2.4** ([2]) Suppose that \( \mathbb{T} \) is an open set in \( \mathbb{R}^n \) and \( 0 < \alpha \leq 1 \). A function \( v: \mathbb{T} \to \mathbb{R} \) is uniformly Hölder continuous with exponent \( \alpha \) in \( \mathbb{T} \) if the quantity
\[
[v]_{\alpha, \mathbb{T}} = \sup_{t,s \in \mathbb{T}, t \neq s} \frac{|v(t) - v(s)|}{|t - s|^{\alpha}}
\]
is finite. A function \( v: \mathbb{T} \to \mathbb{R} \) is locally uniformly Hölder continuous with exponent \( \alpha \) in \( \mathbb{T} \) if \([v]_{\alpha, \mathbb{T}'}\) is finite for every \( \mathbb{T}' \subset \mathbb{T} \). We denote by \( C^{0,\alpha}(\mathbb{T}) \) the space of locally uniformly Hölder continuous functions with exponent \( \alpha \) in \( \mathbb{T} \). If \( \mathbb{T} \) is bounded, we denote by \( C^{0,\alpha}(\overline{\mathbb{T}}) \) the space of uniformly Hölder continuous functions with exponent \( \alpha \) in \( \mathbb{T} \).

The quantity \([v]_{\alpha, \mathbb{T}}\) is a semi-norm, but it is not a norm since it equals to zero for constant functions. The space \( C^{0,\alpha}(\overline{\mathbb{T}}) \), where \( \mathbb{T} \) is bounded, is a Banach space with respect to the norm
\[
\|v\|_{C^{0,\alpha}(\overline{\mathbb{T}})} = \sup_{\mathbb{T}} |v| + [v]_{\alpha, \mathbb{T}}.
\]
In the present article we deal with a generalization of the semi-norm \([v]_{a,T}\) in the space \(C^{0,a}(\mathbb{T})\).

Let’s consider the quantity

\[
[v]_{q,p,T} = \sup_{t,s \in \mathbb{T}, t \neq s} \frac{|v(t) - v(s)|}{q(\rho(t,s))},
\]

where \(\rho\) is a metric in the space \(\mathbb{T}\) and \(q = \{q(t), t \in \mathbb{T}\}\) is a modulus of continuity such that \(\exists \alpha \in (0,1) \forall t, s \in \mathbb{T}, t \neq s\): \(q(\rho(t,s)) \leq |t - s|^\alpha\). If the quantity \([v]_{q,p,T}\) is finite then the quantity \([v]_{a,T}\) is also finite. Therefore if \([v]_{q,p,T}\) is finite for every \(\mathbb{T}' \subset \mathbb{T}\) and the space \(\mathbb{T}\) is bounded then \(v \in C^{0,a}(\mathbb{T})\).

**Definition 2.5 ([3])** Let \((\mathbb{T}, \rho)\) be a metric space. The metric massiveness \(N(u) = N_{(\mathbb{T}, \rho)}(u)\) is the minimal number of closed balls (defined with respect to the metric \(\rho\)) of radius \(u\) that cover \(\mathbb{T}\).

Let us give some properties of the metric massiveness.

**Lemma 2.1 ([3])** The following statements hold:

1) For any \(\varepsilon > 0\) we have \(N_{(\mathbb{T}, \rho)}(\varepsilon) \geq 1\). In this case if \(\varepsilon \geq \text{diam} \mathbb{T} = \sup_{t,s \in \mathbb{T}} \rho(t,s)\), then \(N(\varepsilon) = 1\).

2) The function \(N_{(\mathbb{T}, \rho)}(\varepsilon)\) is right continuous and non-decreasing as \(\varepsilon\) decreases.

3) A space \(\mathbb{T}\) contains a finite number of points if and only if \(\sup_{\varepsilon > 0} N_{(\mathbb{T}, \rho)}(\varepsilon) < \infty\).

Let us formulate the theorem on moduli of continuity of \(L_p(\Omega)\) processes, defined on a compact space.

**Theorem 2.2 ([22])** Let \((\mathbb{T}, \rho)\) be a metric compact space. Consider a separable random process \(X = \{X(t), t \in \mathbb{T}\}\) belonging to the space \(L_p(\Omega), 1 \leq p < \infty\). Suppose that there is a monotonically increasing continuous function \(\sigma = \{\sigma(h), h \geq 0\}\) such that \(\sigma(h) > 0\) as \(h > 0\), \(\sigma(0) = 0\) and the following inequality holds

\[
\sup_{\rho(t,s) \leq h} \|X(t) - X(s)\|_{L_p} \leq \sigma(h).
\]

Let \(N(\varepsilon) = N_{\rho}(\mathbb{T}, \varepsilon)\) be a metric massiveness of the space \((\mathbb{T}, \rho)\). Consider \(\varepsilon_0 = \sigma^{-1}\left(\sup_{t,s \in \mathbb{T}} \rho(t,s)\right)\), where \(\sigma^{-1}(v)\) is the inverse function of the function \(\sigma(v)\), and

\[
g(\varepsilon) = \int_0^{\sigma(\varepsilon_0)} \left(N(\sigma^{-1}(r))\right)^{4/p} dr < \infty, \quad \varepsilon > 0.
\]

Then for \(x > 0, \varepsilon \in (0, \varepsilon_0)\) the following inequality holds true

\[
P\left\{ \sup_{0 < \rho(t,s) \leq \varepsilon} \frac{|X(t) - X(s)|}{(6 + 4\sqrt{2})B^{2/p} f(\rho(t,s)) + (5 + 2\sqrt{6})B^{4/p} g(\rho(t,s))} > x \right\} \leq \frac{2B^2 + B}{B^2 - 1} N(\varepsilon) \cdot x^{p'},
\]

where \(B > 1\) is some number, \(f(\varepsilon) = \int_0^{\sigma(\varepsilon)} \left(N(\sigma^{-1}(r))\right)^{2/p} dr, \varepsilon > 0\).

**Corollary 2.3 ([22])** Let the assumptions of Theorem 2.2 hold true. Then for \(y > 0, \varepsilon \in (0, \varepsilon_0)\) the following inequality holds:
Consider the space $\mathbb{T} = [a, b], 0 \leq a < b < \infty$ with the metric $\rho(t, s) = |t - s|, \ t, s \in [a, b]$. Then

$$\varepsilon_0 = \sigma^{(-1)} \left( \sup_{t, s \in \mathbb{T}} \rho(t, s) \right) = \sigma^{(-1)}(b - a)$$

and the following Corollary holds true.

**Corollary 2.4** Under assumptions of Theorem 2.2 and according to Corollary 2.3, for $y > 0$ and $\varepsilon \in \left( 0, \sigma^{(-1)}(b - a) \right)$ the following inequality holds:

$$P \left\{ \sup_{0 < \rho(t, s) \leq \varepsilon} \frac{|X(t) - X(s)|}{(6 + 4\sqrt{2})f(\rho(t, s)) + (5 + 2\sqrt{6})g(\rho(t, s))} > y \right\} \leq \frac{C_0}{N(\varepsilon) \cdot y^p},$$

where $C_0 = \frac{\sqrt{B_0 + b_0} \sigma^{(1)}(\varepsilon)}{b^{2} - 1} \cdot \frac{\sqrt{B_0 + b_0} \sigma^{(1)}(\varepsilon)}{b^{2} - 1}$, $B_0 = \frac{\sqrt{B_0 + b_0} \sigma^{(1)}(\varepsilon)}{b^{2} - 1}$, $0 < \rho(t, s) \leq \varepsilon$.

$$f(\varepsilon) = \int_0^\varepsilon \left( N(\sigma^{(-1)}(r)) \right)^{2/p} dr \quad \text{and} \quad g(\varepsilon) = \int_0^\varepsilon \left( N(\sigma^{(-1)}(r)) \right)^{4/p} dr.$$

### 3. Main Results

Now we formulate and prove the main result – the theorem on moduli of continuity of $L_p(\Omega)$ processes, defined on an infinite interval $[0, \infty)$, which is based on Theorem 2.2.

**Theorem 3.1** Let $[0, \infty) = \bigcup_{i=0}^{\infty} A_i$, where $A_i = [a_i, a_{i+1}], \ {a_i, i = 0, 1, ...}$ is an increasing sequence, $a_0 = 0$. Denote $\alpha_i = a_{i+1} - a_i, \ D_i = [a_i, a_{i+1} + \theta], \ \theta \in \left( 0, \min_{i \geq 0} \alpha_i \right)$. Consider a separable random process $X = \{X(t), \ t \in [0, \infty)\}$ belonging to the space $L_p(\Omega), 1 \leq p < \infty$. Suppose that there are monotonically increasing continuous functions $\sigma_i = \{\sigma_i(h), \ h \geq 0\}$ such that $\sigma_i(0) = 0, \ i = 0, 1, ...$ and $\forall i = 0, 1, ...$ the following inequality holds

$$\sup_{t \in \mathbb{T}} \|X(t) - X(s)\|_{L_p} \leq \sigma_i(h), \quad 0 < h < \alpha_i + \theta. \tag{1}$$

Let $N_i(\varepsilon), i = 0, 1, ...$ be metric massiveness for $D_i, \ i = 0, 1, ...$ with metric $\rho(t, s) = |t - s|$. Let also

$$\varepsilon_0 = \min_{i \geq 0} \left\{ \sigma_i^{(-1)} \left( \sup_{t, s \in D_i} \rho(t, s) \right) \right\} = \min_{i \geq 0} \left\{ \sigma_i^{(-1)}(\alpha_i + \theta) \right\},$$

where $\sigma_i^{(-1)}(h)$ are inverse functions to functions $\sigma_i(h), \ i = 0, 1, ...,$ and
where

\[
g_i(\varepsilon) = \int_0^\sigma(\varepsilon) \left( N_i \left( \sigma_i^{-1}(r) \right) \right)^{4/p} dr < \infty, \quad \varepsilon > 0;
\]

\[
f_i(\varepsilon) = \int_0^\sigma(\varepsilon) \left( N_i \left( \sigma_i^{-1}(r) \right) \right)^{2/p} dr, \quad \varepsilon > 0.
\]

Denoting \( z_i(t, s) = (6 + 4\sqrt{2})f_i(|t - s|) + (5 + 2\sqrt{6})g_i(|t - s|), \) \( t, s \in D_i \) and \( z(t, s) \) is such function that

\[
z(t, s) = \{ z_i(t, s) \mid t, s \in A_i \cup \text{min}\{t, s\} \in A_i, \text{max}\{t, s\} \in A_{i+1} \},
\]

we obtain that for all \( y > 0, \varepsilon \in (0, \min\{\varepsilon_0, \theta\}) \) and \( \theta > \varepsilon \) under the condition \( \sum_{i=0}^\infty \frac{1}{\alpha_i} < \infty \) the following inequality holds true:

\[
P \left\{ \sup_{0 < |t - s| \leq \varepsilon} \frac{|X(t) - X(s)|}{z(t, s)} > y \right\} \leq \frac{2C_0\varepsilon}{y\delta} \sum_{i=0}^\infty \frac{1}{\alpha_i + \varepsilon},
\]

where \( C_0 = \frac{(2\beta^2 + B_0)B_2}{B_0 - 1} \), \( B_0 = \frac{\sqrt{33}}{4} \cos \left( \frac{1}{3} \arctan \left( \frac{8\sqrt{41}}{37} \right) \right) - \frac{1}{8} \).

**Proof:** According to Theorem 2.2 and Corollary 2.4, \( \forall i = 0, 1, \ldots, y > 0 \) and \( \varepsilon \in \left( 0, \alpha_i^{-1}(\alpha_i + \theta) \right) \) the following inequality holds

\[
P \left\{ \sup_{0 < |t - s| \leq \varepsilon} \frac{|X(t) - X(s)|}{\left( 6 + 4\sqrt{2} \right)f_i(|t - s|) + \left( 5 + 2\sqrt{6} \right)g_i(|t - s|)} > y \right\} \leq \frac{C_0}{N_i(\varepsilon) \cdot y\delta}, \tag{2}
\]

where \( C_0 = \frac{(2\beta^2 + B_0)B_2}{B_0 - 1} \), \( B_0 = \frac{\sqrt{33}}{4} \cos \left( \frac{1}{3} \arctan \left( \frac{8\sqrt{41}}{37} \right) \right) - \frac{1}{8} \).

\[
f_i(\varepsilon) = \int_0^\sigma(\varepsilon) \left( N_i \left( \sigma_i^{-1}(r) \right) \right)^{2/p} dr \quad \text{and} \quad g_i(\varepsilon) = \int_0^\sigma(\varepsilon) \left( N_i \left( \sigma_i^{-1}(r) \right) \right)^{4/p} dr.
\]

Since \( \forall i = 0, 1, \ldots: \)

\[
\frac{\alpha_i + \theta}{2\varepsilon} \leq N_i(\varepsilon) \leq \frac{\alpha_i + \theta}{2\varepsilon} + 1,
\]

then \( \forall i = 0, 1, \ldots: \)

\[
\frac{1}{N_i(\varepsilon)} \leq \frac{2\varepsilon}{\alpha_i + \theta}. \tag{3}
\]

Inequalities (2) and (3) imply that for \( \theta \in \left( 0, \min\{\alpha_i\} \right), y > 0 \) and \( \varepsilon \in (0, \min\{\varepsilon_0, \theta\}) \) under the condition \( \sum_{i=0}^\infty \frac{1}{\alpha_i} < \infty \) the following inequality holds

\[
P \left\{ \sup_{0 < |t - s| \leq \varepsilon} \frac{|X(t) - X(s)|}{z(t, s)} > y \right\} = \frac{2\varepsilon}{\alpha_i + \theta} \sum_{i=0}^\infty \frac{1}{\alpha_i + \theta} \cdot \frac{2\varepsilon}{\alpha_i + \theta} \sum_{i=0}^\infty \frac{1}{\alpha_i + \theta} \sum_{i=0}^\infty \frac{1}{\alpha_i + \theta}.
\]
\[
\begin{align*}
&\leq \sum_{i=0}^{\infty} P \left\{ \sup_{\substack{0 < |t-s| \leq \varepsilon \\ t,s \in D_i}} \frac{|X(t) - X(s)|}{z(t,s)} > y \right\} \\
&\leq \sum_{i=0}^{\infty} P \left\{ \sup_{\substack{0 < |t-s| \leq \varepsilon \\ t,s \in D_i}} \frac{|X(t) - X(s)|}{z_i(t,s)} > y \right\} \\
&\leq \sum_{i=0}^{\infty} C_0 N_i(\varepsilon) \cdot y^p \\
&\leq \frac{C_0}{y^p} \sum_{i=0}^{\infty} \frac{2\varepsilon}{\alpha_i + \theta} = \frac{2C_0\varepsilon}{y^p} \sum_{i=0}^{\infty} \frac{1}{\alpha_i + \theta}.
\end{align*}
\]

Since \(\theta > \varepsilon\), then, substituting \(\varepsilon\) instead of \(\theta\) in the last inequality, we obtain the statement of the theorem. \(\square\)

**Remark 4.1** Let assumptions of Theorem 3.1 hold true. If \(\forall i = 0, 1, \ldots : \sigma_i(h) = \sigma(h)\) then the inequality (1) takes the following form:

\[
\sup_{|t-s| \leq h \atop t,s \in D_i} \|X(t) - X(s)\|_{L_p} \leq \sigma(h), \quad 0 < h < \alpha_i + \theta, \ i = 0, 1, \ldots
\]

Therefore for all \(i = 0, 1, \ldots\) and \(\varepsilon > 0\):

\[
\begin{align*}
&f_i(\varepsilon) = \int_{\sigma(\varepsilon)}^{\sigma^{(-1)}(r)} \left( N_i(\sigma^{(-1)}(r)) \right)^{2/p} dr, \\
&g_i(\varepsilon) = \int_{\sigma(\varepsilon)}^{\sigma^{(-1)}(r)} \left( N_i(\sigma^{(-1)}(r)) \right)^{4/p} dr,
\end{align*}
\]

and since \(\sigma^{(-1)}(h)\) is a non-decreasing function we have that

\[
\varepsilon_0 = \min_{i \geq 0} \{\sigma^{(-1)}(\alpha_i + \theta)\} = \sigma^{(-1)}(\min_{i \geq 0} \alpha_i + \theta).
\]

**Example 4.1** Let \(\forall i = 0, 1, \ldots\) functions \(\sigma_i(h) = \sigma(h) = dh^\beta, h, \beta, d > 0\).

The inverse function to the function \(\sigma(h)\) is \(\sigma^{(-1)}(h) = \sqrt[\beta]{\frac{h}{d}}\). Therefore, functions \(f_i(\varepsilon)\) and \(g_i(\varepsilon), \ i = 0, 1, \ldots\), take the following form:

\[
\begin{align*}
&f_i(\varepsilon) = \int_{\sigma(\varepsilon)}^{\sigma^{(-1)}(r)} \left( N_i(\sigma^{(-1)}(r)) \right)^{2/p} dr = \int_{\sigma(\varepsilon)}^{\sigma^{(-1)}(r)} \left( N_i \left( \sqrt[\beta]{\frac{r}{d}} \right) \right)^{2/p} dr;
\end{align*}
\]
Indeed, let estimate and evaluate integrals. According to Theorem 3.1 and Remark 4.1, for \( \varepsilon \in (0, \min \{ \varepsilon_0, \theta \}) \), \( \varepsilon_0 = \sigma^{(-1)} \left( \min_{i \geq 0} \alpha_i + \theta \right) = \frac{1}{\beta^4} \sqrt{\min_{i \geq 0} \alpha_i + \theta} \), \( \theta \in \left( 0, \min_{i \geq 0} \alpha_i \right) \) and \( \forall y > 0 \) under the condition \( \sum_{i=0}^{\infty} \frac{1}{\alpha_i} < \infty \) the following inequality holds:

\[
P \left\{ \sup_{0 < |t-s| \leq \varepsilon} \frac{|X(t) - X(s)|}{z(t, s)} > y \right\} \leq \frac{2C_0}{y^p} \sum_{i=0}^{\infty} \frac{1}{\alpha_i + \varepsilon},
\]

where \( z(t, s) = \{ z_i(t, s) : t, s \in A_i, \min\{t, s\} \in A_i, \max\{t, s\} \in A_i+1 \} \),

\[
z_i(t, s) = (6 + 4\sqrt{2}) \int_0^1 \left( N_i \left( \frac{\beta }{\sqrt{d}} \right) \right)^{2/p} dr + (5 + 2\sqrt{6}) \int_0^1 \left( N_i \left( \frac{\beta }{\sqrt{d}} \right) \right)^{4/p} dr,
\]

\[
C_0 = \frac{(2B_2^3 + B_0)B_0}{b_2 - 1}, B_0 = \sqrt{3} \cos \left( \frac{1}{3} \arctan \left( \frac{8\sqrt{47}}{37} \right) \right) - \frac{1}{8}.
\]

Since the metric massiveness \( N_i(u) \) denotes the minimal number of elements in an \( u \)-covering of the interval \( [a_i, a_{i+1} + \theta] \) then

\[
\frac{\alpha_i + \theta}{2u} \leq N(u) \leq \frac{\alpha_i + \theta}{2u} + 1.
\]

Therefore, using the inequality for the metric massiveness

\[
N_i \left( \frac{\beta }{\sqrt{d}} \right) = N_i \left( \sigma^{(-1)}(r) \right) \leq \frac{\alpha_i + \theta}{2\sigma^{(-1)}(r)} + 1 = \frac{\alpha_i + \theta}{2} \left( \frac{\sqrt{d}}{\sqrt{r}} + 1 \right) = \frac{\alpha_i + \theta}{2} \left( \frac{\sqrt{d}}{\sqrt{r}} + 1 \right),
\]

we evaluate functions \( f_i(\varepsilon) \) and \( g_i(\varepsilon) \):

\[
f_i(\varepsilon) = \int_0^{\alpha_i + \theta} \left( \int_0^{\frac{\beta }{\sqrt{d}}} \left( N_i \left( \frac{\beta }{\sqrt{d}} \right) \right)^{2/p} dr \right) d\varepsilon^\beta =: I_{1,i},
\]

\[
g_i(\varepsilon) = \int_0^{\alpha_i + \theta} \left( \int_0^{\frac{\beta }{\sqrt{d}}} \left( N_i \left( \frac{\beta }{\sqrt{d}} \right) \right)^{4/p} dr \right) d\varepsilon^\beta =: I_{2,i}.
\]

Let estimate and evaluate integrals \( I_{1,i} \) and \( I_{2,i} \). For the first one the following inequality holds true:

\[
I_{1,i} \leq \int_0^{\alpha_i + \theta} \left( \frac{\sqrt{d}}{\frac{\beta }{\sqrt{d}}} \right)^{2/p} dr.
\]

Indeed,
\[
\frac{\alpha_i + \theta}{2} \sqrt[\beta]{\frac{d}{r}} + 1 \leq \frac{c_1(\varepsilon)}{\beta} \Rightarrow \frac{(\alpha_i + \theta) \sqrt[\beta]{d}}{2} + \frac{\beta}{\sqrt[r]{r}} \leq c_1(\varepsilon),
\]

then, since \( t \in (0, d e^\beta) \), we obtain:

\[
c_1(\varepsilon) \geq \frac{(\alpha_i + \theta) \sqrt[\beta]{d}}{2} + \frac{\beta}{\sqrt[r]{r}} = \frac{(\alpha_i + \theta + \varepsilon)}{2}.
\]

Therefore

\[
\int_0^{d e^\beta} \left( \frac{\alpha_i + \theta}{2} \sqrt[\beta]{\frac{d}{r}} + 1 \right)^{2/p} dr \leq \frac{d e^\beta}{(2/p)} \left( \frac{\alpha_i + \theta + \varepsilon}{2} \right)^{2/p} \int_0^{d e^\beta} r^{(2/p)} dr.
\]

The last integral is finite if the condition \( \frac{2}{\beta p} < 1 \Leftrightarrow \beta > \frac{2}{p} \) is satisfied. Thus, for \( \beta > \frac{2}{p} \) we evaluate the integral \( I_{1,i} \):

\[
I_{1,i} \leq \frac{\beta d}{\beta p - 4} \left( \frac{\alpha_i + \theta}{2} + \varepsilon \right)^{4/p} \frac{\beta p - 4}{\varepsilon p}.
\]

Analogously for the integral \( I_{2,i} \) the following inequality holds for \( \beta > \frac{4}{p} \):

\[
I_{2,i} \leq \frac{\beta d}{\beta p - 4} \left( \frac{\alpha_i + \theta}{2} + \varepsilon \right)^{4/p} \frac{\beta p - 4}{\varepsilon p}.
\]

Therefore in accordance with Theorem 3.1 and Remark 4.1, for \( \varepsilon \in \left( 0, \min_{i \geq 0} \frac{1}{\alpha_i} \right) \), \( \beta > \frac{4}{p} \), \( \theta \in \left( 0, \min \alpha_i \right) \) and \( \forall y > 0 \) under the condition \( \sum_{i=0}^{\infty} \frac{1}{\alpha_i} < \infty \) the following inequality holds

\[
P \left( \sup_{0 < |t-s| \leq \varepsilon} \left| \frac{X(t) - X(s)}{z(t,s)} \right| > y \right) \leq \frac{2C_0}{y^p} \sum_{i=0}^{\infty} \frac{1}{\alpha_i + \varepsilon}
\]

where \( z(t,s) = \{ z_i(t,s) \mid t, s \in A_i \forall \min \{t,s\} \in A_i, \max \{t,s\} \in A_{i+1} \} \),

\[
z_i(t,s) = \beta d \left( \frac{\alpha_i + \varepsilon}{2} + |t-s| \right)^{2/p} \left( \frac{6 + 4\sqrt{2}}{\beta p - 2} |t-s|^{\frac{2}{\beta p - 2}} + \frac{5}{\beta p - 4} \left( \frac{\alpha_i + \varepsilon}{2} + |t-s| \right)^{2/p} \right), \quad |t-s|^{\frac{\beta p - 4}{p}}
\]

\[
C_0 = \left( \frac{2B_0 + B_3}{B_5 - 1} \right), B_0 = \frac{\sqrt{33}}{4} \cos \left( \frac{1}{3} \arctan \left( \frac{8 \sqrt{41}}{37} \right) \right), \quad \frac{1}{8}
\]

**Example 4.2** Let a stationary process \( X = \{ X(t), t \in [0,\infty) \} \) belonging to the space \( L_p(\Omega), 1 \leq p < \infty \), with \( \mathbb{E}X(t) = 0 \) and \( \mathbb{E}X^2(t) = 1 \) satisfy the assumptions of Theorem 3.1 and \( \forall i = 0,1,... \) functions

\[
\sigma_i(h) = \sigma(h) = dh^\beta, \quad h, \beta, d > 0
\]

as in Example 4.1. Consider a process \( Y(t) = \frac{X(t)}{c(t)} \), where \( c(t), t \in [0,\infty) \) is some monotonically increasing function such that
\[c(\alpha_i) \geq (\alpha_i + 3\theta)^{\frac{4}{p}}, \quad i = 0, 1, \ldots \]  

(4)

and \(\forall t, s \in [0, \infty), \ |t - s| < \max_{i \geq 0} \alpha_i + \theta\) there exist such \(b > 0\) and \(\gamma \in (0, \frac{4}{p} + \beta)\) that the following inequality holds

\[|c(t) - c(s)| \leq b|t - s|^\gamma.\]  

(5)

Since \(\forall t, s \in [0, \infty), \ |t - s| < \max_{i \geq 0} \alpha_i + \theta:\)

\[\|Y(t) - Y(s)\|_{L_p} = \left\| \frac{X(t) - X(s)}{c(t)} + \frac{X(s)}{c(t)} - \frac{X(s)}{c(s)} \right\|_{L_p}
\leq \frac{1}{|c(t)|} \cdot \|X(t) - X(s)\|_{L_p} + \|X(s)\|_{L_p} \cdot \left| \frac{1}{c(t)} - \frac{1}{c(s)} \right|
\leq \frac{d|t - s|^\beta}{|c(t)|} + \frac{|c(t) - c(s)|}{c(t)c(s)}.
\]

then implying inequality (5) we have \(\forall t, s \in [0, \infty), \ |t - s| < \max_{i \geq 0} \alpha_i + \theta:\)

\[\|Y(t) - Y(s)\|_{L_p} \leq \frac{d|t - s|^\beta}{|c(t)|} + \frac{|c(t) - c(s)|}{c(t)c(s)}
= |t - s|^\beta \cdot \frac{1}{|c(t)|} \left( d + \frac{b|t - s|^\gamma - \beta}{c(s)} \right).
\]

Thus, implying inequality (4), \(\forall i = 0, 1, \ldots, h \in (0, \alpha_i + \theta)\):

\[
\sup_{t, s \in \mathcal{D}_i} \|Y(t) - Y(s)\|_{L_p} \leq \frac{h^\beta}{c(\alpha_i)} \left( d + \frac{b(\alpha_i + \theta)^{\gamma - \beta}}{c(\alpha_i)} \right)
\leq \frac{d + b}{c(\alpha_i)} . h^\beta.
\]

Therefore, as in Example 4.1, we obtain that under the condition \(\beta > \frac{4}{p}\) the following holds

\[
f_i(\varepsilon) \leq \frac{\beta p(d + b)}{2^2(p(\beta p - 2))} \varepsilon \cdot \frac{\alpha_i + \theta + 2\varepsilon}{c(\alpha_i)} ^ {2/p} \leq \frac{\beta p(d + b)}{2^2(p(\beta p - 2))} \varepsilon \cdot \frac{\alpha_i + \theta}{c(\alpha_i)} ^ {2/p} ;
\]

\[
g_i(\varepsilon) \leq \frac{\beta p(d + b)}{2^4(p(\beta p - 4))} \varepsilon \cdot \frac{\alpha_i + \theta + 2\varepsilon}{c(\alpha_i)} ^ {4/p} \leq \frac{\beta p(d + b)}{2^4(p(\beta p - 4))} \varepsilon \cdot \frac{\alpha_i + \theta}{c(\alpha_i)} ^ {4/p} .
\]

Finally, according to Theorem 3.1 and Remark 4.1, for \(\varepsilon \in \left(0, \min \left\{ \frac{1}{\beta + \beta \min_{i \geq 0} \alpha_i + \theta, \theta} \right\} \right), \beta > \frac{4}{p}\)

\(\theta \in \left(0, \min_{i \geq 0} \alpha_i \right)\) and \(\forall y > 0\) under the condition \(\sum_{i=0}^{\infty} \frac{1}{\alpha_i} < \infty\) the following inequality holds

\[P \left\{ \sup_{0 < |t - s| \leq \varepsilon, t, s \in [0, \infty)} \frac{|Y(t) - Y(s)|}{z(t, s)} > y \right\} \leq \frac{2C_0}{y^p} \sum_{i=0}^{\infty} \frac{1}{\alpha_i + \varepsilon}.
\]
where \( C_0 = \frac{2B_0^2 + B_0B_1}{B_0^2 - 1} \), \( B_0 = \sqrt[3]{33} \cos \left( \frac{1}{3} \arctan \left( \frac{8\sqrt{37}}{37} \right) \right) - \frac{1}{3} \) and

\[
z(t, s) = \frac{\beta p(d + b)}{2^{2/p}} \left( 6 + 4\sqrt{2} \frac{|t - s|^{2/p}}{\beta p - 2} + \frac{5 + 2\sqrt{6}}{2^{2/p}(\beta p - 4)} \right) \cdot |t - s|^{\frac{\beta p - 4}{p}}.
\]

For example, let \( p = 6, \alpha_i = 2^i, i = 0, 1, ..., d = 1, \beta = \frac{5}{\epsilon}, \theta = 0.5 \) and the function \( c(t) = (t + 2.5)^\gamma \), \( \gamma \in \left( \frac{2}{3}, \frac{3}{2} \right) \). Then \( a_i = 2^i - 1, i = 0, 1, ... \), the function \( c(t) \) satisfies the condition (5) with \( b = 1 \) and \( \epsilon \in (0, 0.5) \). Therefore the function

\[
z(t, s) \leq 5 \cdot 2^{1/3} \cdot \left( 7 + \frac{4\sqrt{2}}{3} + 2\sqrt{6} \right) \cdot |t - s|^{1/6} < 86.838 \cdot |t - s|^{1/6}
\]

and, according to Theorem 3.1 and Remark 4.1, for \( \epsilon \in (0, 0.5) \) and \( \forall \gamma > 0 \) the following inequality holds

\[
P \left\{ \sup_{0 \leq |t - s| \leq \epsilon, t, s \in [0, \infty)} \frac{|Y(t) - Y(s)|}{|t - s|^{1/6}} > 86.838 \gamma \right\} < \frac{75.912}{\gamma^6}.
\]

Moreover, for \( \epsilon \in (0, 0.5) \) the following holds

\[
\sup_{0 \leq |t - s| \leq \epsilon, t, s \in [0, \infty)} \frac{|Y(t) - Y(s)|}{|t - s|^{1/6}} < 86.838 \gamma
\]

with probability greater than 0.95, if \( \gamma \geq 3.391 \).

4. Conclusion

In this article we obtain estimates for distributions of semi-norms of sample functions of \( L_p(\Omega) \) processes, defined on an infinite interval \([0, \infty)\), in Hölder spaces in general case and for particular functions \( \sigma_i(h) \) as examples.

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