Columbia International Publishing
Journal of Applied Mathematics and Statistics
(2015) Vol. 2 No. 1 pp. 21-42
doi:10.7726/jams.2015.1003
Research Article

Minimax Interpolation Problem for Harmonizable Stable Sequences with Noise Observations

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Received 22 July 2015; Published online 26 December 2015

Abstract

We consider the problem of optimal linear estimation of the functional $A_N\xi=\sum_{j=0}^{N}a_j\xi_j$ that depends on the unknown values $\xi_j, j=0,1,\dots,N$, of a random sequence $\{\xi_k, k\in\mathbf{Z}\}$ from observations of the sequence $\{\xi_k+\eta_k, k\in\mathbf{Z}\}$ at points of time $k\in\mathbf{Z}$, $\{0,1,\dots,N\}$, where $\{\xi_k, k\in\mathbf{Z}\}$ and $\{\eta_k, k\in\mathbf{Z}\}$ are mutually independent harmonizable symmetric α-stable random sequences which have the spectral densities $f(\theta)$ and $g(\theta)$ satisfying the minimality condition. The problem is investigated under the condition of spectral certainty as well as under the condition of spectral uncertainty. Formulas for calculation the value of the error and the spectral characteristic of the optimal linear estimate of the functional are derived under the condition of spectral certainty, where spectral densities of the sequences are exactly known. In the case of spectral uncertainty, where spectral densities of the sequences are not exactly known while a class of admissible spectral densities is given, relations that determine the least favorable spectral densities and the minimax spectral characteristic are derived.

Keywords: Harmonizable sequence; Optimal linear estimate; Minimax-robust estimate; Least favorable spectral density; Minimax spectral characteristic

2000 Mathematics Subject Classification: Primary: 60G10, 60G25, 60G35, Secondary: 62M20, 93E10, 93E11

1. Introduction

The classical methods of finding solutions to extrapolation, interpolation and filtering problems for stationary stochastic processes and sequences were developed by Kolmogorov (see selected works by Kolmogorov (1992)), Wiener (see book by Wiener (1966)), Yaglom (see, for example, books by

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Yaglom (1987a, 1987b). The problem of estimation of the unknown values of harmonizable random sequences and processes were investigated in papers by Cambanis (1983), Cambanis and Soltani (1984), Hosoya (1982). The interpolation problem for harmonizable symmetric α-stable random sequences was investigated in papers by Weron (1985) and Pourahmadi (1984). Most of results concerning estimation of the unknown (missed) values of stochastic processes are based on the assumption that spectral densities of processes are exactly known. In practice, however, complete information on the spectral densities is impossible in most cases. In such situations one finds parametric or nonparametric estimates of the unknown spectral densities. Then the classical estimation method is applied under the assumption that the estimated densities are true. This procedure can result in significant increasing of the value of error as Vastola and Poor (1983) have demonstrated with the help of some examples. This is a reason to search estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximal value of the error. A survey of results in minimax (robust) methods of data processing can be found in the paper by Kassam and Poor (1985). The paper by Grenander (1957) should be marked as the first one where the minimax extrapolation problem for stationary processes was formulated and solved. Later Franke and Poor (Franke and Poor, 1984; Franke, 1985) applied the convex optimization methods to investigation the minimax-robust extrapolation and interpolation problems. In papers by Moklyachuk (1990 – 2015) the minimax-robust extrapolation, interpolation and filtering problems are studied for stationary processes. In the book by Moklyachuk and Masyutka (2012) results of investigation the minimax-robust extrapolation, interpolation and filtering problems for vector-valued stationary processes and sequences are described. In the paper by Dubovetska et al. (2012) the problem of minimax-robust interpolation is investigated for another generalization of stationary processes – periodically correlated sequences. In the papers by Dubovetska and Moklyachuk (2013 – 2014) and book by Golichenko and Moklyachuk (2014) the minimax-robust extrapolation, interpolation and filtering problems for periodically correlated processes are investigated. The minimax-robust extrapolation, interpolation and filtering problems for stochastic sequences and random processes with n-th stationary increments are investigated by Luz and Moklyachuk (Luz and Moklyachuk, 2012 – 2015b; Moklyachuk and Luz, 2013).

In this paper the problem of optimal estimation is investigated for the linear functional $A_N\xi=\sum_{j=0}^{N}a_j\xi_j$ that depends on the unknown values of a random sequence $\xi_j, j=0,1,\dots,N$, from observations of the sequence $\{\xi_k+\eta_k, k\in\mathbf{Z}\}$ at points of time $k\in\mathbf{Z}$, $\{0,1,\dots,N\}$, where $\{\xi_k, k\in\mathbf{Z}\}$ *and* $\{\eta_k, k\in\mathbf{Z}\}$ are mutually independent harmonizable symmetric α-stable random sequences which have spectral densities $f(\theta)>0$ *and* $g(\theta)>0$ satisfying the minimality condition. The problem is investigated under the condition of spectral certainty as well as under the condition of spectral uncertainty. Formulas for calculation the value of the error and the spectral characteristic of the optimal linear estimate of the functional are derived under the condition of spectral certainty, where spectral densities of the sequences are exactly known. In the case where spectral densities of the sequences are not exactly known while sets of admissible spectral densities are available, relations which determine the least favorable densities and the minimax-robust spectral characteristics for different classes of spectral densities are derived.

2. Harmonizable Symmetric α-Stable Random Sequence

Definition 2.1 (symmetric α-stable random variable) *A real random variable ξ is said to be symmetric α-stable, $S\alpha S$, if its characteristic function has the form $E\exp(it\xi)=\exp(it\xi)=\exp(-c|t|^{\alpha})$ for some $c>0$ and $0<\alpha\le 2$. The real random variables $\xi_1,\xi_2,\dots,\xi_n$ are jointly $S\alpha S$ if all linear combinations $\sum_{k=1}^{n}a_k\xi_k$ are $S\alpha S$, or equivalently if the characteristic function of $\vec{\xi}=(\xi_1,\xi_2,\dots,\xi_n)$ is of the form $\vec{\varphi}_{\vec{\xi}}=E\exp\left(i\sum t_k\xi_k\right)=\exp\left(-\int\left|\sum t_k x_k\right|^{\alpha}d\Gamma_{\vec{\xi}}(\vec{x})\right)$, where $t_1,t_2,\dots,t_n$ are real numbers and $\Gamma_{\vec{\xi}}(\vec{x})$ is a symmetric measure defined on the unit sphere $S_n\in R^n$ (Cambanis, 1983).*

Definition 2.2 (symmetric α-stable stochastic sequence) *A stochastic sequence $\{\xi_k, k\in\mathbb{Z}\}$ is called symmetric α-stable, $S\alpha S$, stochastic sequence, if all linear combinations $\sum_{k=1}^{n}a_k\xi_k$ are $S\alpha S$ random variables.*

For jointly $S\alpha S$ random variables $\xi=\xi_1+i\xi_2$ and $\eta=\eta_1+i\eta_2$ the covariation of ξ with η is defined as (Cambanis, 1983)

$$[\xi,\eta]_{\alpha}=\int_{S_4}(x_1+ix_2)(y_1+iy_2)^{<\alpha-1>}\,d\Gamma_{\xi_1,\xi_2,\eta_1,\eta_2}(x_1,x_2,y_1,y_2),$$

where $z^{<\beta>}=|z|^{\beta-1}\overline{z}$ for a complex number z and $\beta>0$. The covariation in general is not symmetric and linear on second argument (Weron, 1985). For jointly $S\alpha S$ random variables ξ,ξ_1,ξ_2,η we have

$$[\xi_1+\xi_2,\eta]_{\alpha}=[\xi_1,\eta]_{\alpha}+[\xi_2,\eta]_{\alpha},$$
$$\left|[\xi,\eta]_{\alpha}\right|\le\|\xi\|_{\alpha}\|\eta\|_{\alpha}^{\alpha-1}, \tag{1}$$

and $\|\xi\|_{\alpha}=[\xi,\xi]_{\alpha}^{1/\alpha}$ is a norm in the linear space of $S\alpha S$ random variables which is equivalent to convergence in probability. It should be noted that $\|\cdot\|_{\alpha}$ is not necessarily the usual L^{α} norm.

Here is the simplest property of the function z^{β}.

Lemma 2.1 *Let z,x,y, $\beta>0$. Then the following properties holds true:*

- $|z|^{<\beta>}=z\cdot z^{<\beta-1>}$,
- $\left||z|^{<\beta>}\right|=|z|^{<\beta>}$,
- if $z^{<\beta>}=v$, then $z=|v|^{<1/\beta>}=v^{(1-\beta)/\beta}\overline{v}$,
- $z^{<1>}=\overline{z}$,
- if $z\ne 0$, then $z^{<\alpha>}z^{<\beta>}=\dfrac{\overline{z}}{|z|}z^{<\alpha+\beta>}$,

- if $z \neq 0$, then $\frac{z^{<\alpha>}}{z^{<\beta>}} = \frac{z}{|z|} z^{<\alpha-\beta>}$,
- $(cz)^{<\alpha>} = c^{\alpha} z^{<\alpha>}, c \in R,$
- $\left(z^{\alpha}\right)^{<\beta>} = \left(z^{\beta}\right)^{<\alpha>}$,
- $\left|z^{<\alpha>}\right|^{\beta} = |z|^{\alpha\beta}$,
- $\left(x+y\right)^{<\alpha>} = \bar{x}\left|x+y\right|^{\alpha-1} + \bar{y}\left|x+y\right|^{\alpha-1}$.

Let $Z = \{Z(t): -\infty < t < \infty\}$ be a complex $S\alpha S$ process with independent increments. The spectral measure of the process Z is defined as $\mu\{(s,t]\} = \|Z(t) - Z(s)\|_{\alpha}^{\alpha}$. The integrals $\int f(t)dZ(t)$ can be defined for all $f \in L^{\alpha}(\mu)$ with the properties (see Cambanis, 1983; Cambanis and Soltani, 1984; Hosoya, 1982):

$$\left\|\int fdZ\right\|_{\alpha}^{\alpha} = \int |f|^{\alpha} d\mu, \ \left[\int fdZ, \int gdZ\right]_{\alpha} = \int f\left(g\right)^{<\alpha-1>} d\mu. \tag{2}$$

Definition 2.3 (Harmonizable symmetric α-stable stochastic sequence)
A $S\alpha S$ stochastic sequence $\{\xi_n, n \in Z\}$ is said to be harmonizable, $HS\alpha S$, if there exists a $S\alpha S$ process $Z = \{Z(\theta): \theta \in [-2\pi, 2\pi]\}$ with independent increments and finite spectral measure μ such that the sequence ξ_n has the spectral representation

$$\xi_n = \int_{-\pi}^{\pi} e^{in\theta} dZ(\theta), n \in \mathbf{Z},$$

and the covariation has the representation

$$[\xi_n, \xi_m]_{\alpha} = \int_{-\pi}^{\pi} e^{i(n-m)\theta} dZ(\theta), \quad n, m \in \mathbf{Z}.$$

Note that a $HS\alpha S$ stochastic sequence is not necessarily stationary even second order stationary, but for $\alpha = 2$ the $HS\alpha S$ sequences are stationary with Gaussian distribution. In this article we consider the case where $1 < \alpha \leq 2$.
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enote by $H(\xi)$ the time domain of the $HS\alpha S$ sequence $\{\xi_n, n \in \mathbf{Z}\}$ which is a closed in the norm $\|\cdot\|_{\alpha}$ linear manifold generated by all values of the $HS\alpha S$ sequence $\{\xi_n, n \in \mathbf{Z}\}$. It follows from the spectral representation of the $HS\alpha S$ sequence $\{\xi_n, n \in \mathbf{Z}\}$ that the mapping $\xi_n \leftrightarrow e^{in\theta}, n \in \mathbf{Z}$, extents to an isomorphism between the spaces $H(\xi)$ and $L^{\alpha}(\mu)$. Under this isomorphism to each $\eta \in H(\xi)$ corresponds a unique $f \in L^{\alpha}(\mu)$ such that $\eta = \int_{-\pi}^{\pi} f(\theta) dZ(\theta)$.

For a closed linear subspace $M \subseteq L^{\alpha}(\mu)$ and $f \in L^{\alpha}(\mu)$, there exists a unique element from M which minimizes the distance to f. This element is called projection of f onto M or the best approximation of f in M. This projection is denoted by $P_M f$ and is uniquely determined by the condition (Singer, 1970)

$$\int_{-\pi}^{\pi} g\left(f - P_M f\right)^{<\alpha-1>} d\mu = 0, \quad g \in M. \tag{3}$$

Similarly, for $HS\alpha S$ stochastic sequence $\{\xi_n, n \in \mathbf{Z}\}$ and a closed linear subspace $H^N(\xi)$ of the space $H(\xi)$ there is a uniquely determined element $\widehat{\xi}_n \in H(\xi)$ which minimizes the distance to ξ_n and is uniquely determined from the condition

$$\left[\eta, \xi_n - \widehat{\xi}_n\right]_\alpha = 0, \quad \eta \in H^N(\xi). \tag{4}$$

From linearity of the covariation with respect to the first argument from this relation we have that

$$\left\|\xi_n - \widehat{\xi}_n\right\|_\alpha^\alpha = \left[\xi_n, \xi_n - \widehat{\xi}_n\right]_\alpha - \left[\widehat{\xi}_n, \xi_n - \widehat{\xi}_n\right]_\alpha = \left[\xi_n, \xi_n - \widehat{\xi}_n\right]_\alpha. \tag{5}$$

This relation plays a fundamental role in the characterization of minimal $HS\alpha S$ stochastic sequences $\{\xi_n, n \in Z\}$ and in finding the best linear interpolator.

3. Interpolation Problem. Projection Approach

Consider the problem of the optimal estimation of the linear

$$A_N\xi = \sum_{j=0}^{N} a_j \xi_j = \int_{-\pi}^{\pi} A_N(e^{i\theta}) dZ(\theta),$$

where

$$A_N(e^{i\theta}) = \sum_{j=0}^{N} a_j e^{ij\theta},$$

that depends on the unknown values $\xi_j, j = 0,1,\ldots,N$, of a random sequence $\{\xi_k, k \in \mathbf{Z}\}$ from observations of the sequence $\{\xi_k + \eta_k, k \in \mathbf{Z}\}$ at points of time $k \in \mathbf{Z}$, $\{0,1,\ldots,N\}$, where $\{\xi_k, k \in \mathbf{Z}\}$ and $\{\eta_k, k \in \mathbf{Z}\}$ are mutually independent harmonizable symmetric α-stable random sequences which have the spectral densities $f(\theta) > 0$ and $g(\theta) > 0$ satisfying the minimality condition (Kolmogorov, 1992; Rozanov, 1967; Salehi, 1979; Pourahmadi, 1984; Weron, 1985) .

$$\int_{-\pi}^{\pi} \left(f(\theta) + g(\theta)\right)^{-1/(\alpha-1)} d\theta < \infty. \tag{6}$$

Denote by $H^N(\xi+\eta)$ the closed in the $\|\cdot\|_\alpha$ norm linear manifold generated by values of the harmonizable symmetric α-stable random sequence $\xi_k + \eta_k, k \in \mathbf{Z}$, $\{0,1,\ldots,N\}$,in the space $H(\xi+\eta)$ generated by all values of the $HS\alpha S$ sequence $\{\xi_k + \eta_k, k \in \mathbf{Z}\}$.

The optimal estimate $\widehat{A}_N\xi$ of the functional $A_N\xi$ is a projection of $A_N\xi$ on the subspace $H^N(\xi+\eta)$ which is determined by the relations

$$\left[\varsigma, A_N\xi - \widehat{A}_N\xi\right]_\alpha = 0, \quad \forall \varsigma \in H^N(\xi+\eta)$$

or, equivalently, by relations

$$\left[\xi_k+\eta_k, A_N\xi-\widehat{A}_N\xi\right]_\alpha=0,\ \ \forall k\in\mathbf{Z}\ ,\ \ \{0,1,\ldots,N\}. \tag{7}$$

It follows from the isomorphism between the spaces $H(\xi+\eta)$ and $L^\alpha(f+g)$ that the optimal estimate $\widehat{A}_N\xi$ of the functional $A_N\xi$ is of the form

$$\widehat{A}_N\xi=\int_{-\pi}^{\pi}h(\theta)\left(Z^\xi(\theta)+Z^\eta(\theta)\right). \tag{8}$$

It is determined by the spectral characteristic $h(\theta)$ of the estimate which is from the subspace $L_N^\alpha(f+g)$ of the $L^\alpha(f+g)$ space generated by functions $e^{ik\theta}, k\in\mathbf{Z}\ ,\ \{0,1,\ldots,N\}$. The spectral characteristic $h(\theta)$ of the optimal estimate satisfies the following equations

$$\int_{-\pi}^{\pi}e^{i\theta k}\left[\left(A_N\left(e^{i\theta}\right)-h(\theta)\right)^{<\alpha-1>}f(\theta)-\left(h(\theta)\right)^{<\alpha-1>}g(\theta)\right]d\theta=0,\ \ k\in\mathbf{Z}\ ,\ \ \{0,1,\ldots,N\}. \tag{9}$$

It follows from these equations that the spectral characteristic $h(\theta)$ of the estimate is determined by the equation

$$\left(A_N\left(e^{i\theta}\right)-h(\theta)\right)^{<\alpha-1>}f(\theta)-\left(h(\theta)\right)^{<\alpha-1>}g(\theta)=\overline{C_N\left(e^{i\theta}\right)}, \tag{10}$$

$$C_N\left(e^{i\theta}\right)=\sum_{i=0}^{N}c_je^{ij\theta},$$

where c_j are unknown coefficients. These unknown coefficients c_j are determined from the condition $h(\theta)\in L_N^\alpha(f+g)$ which gives us the system of equations

$$\int_{-\pi}^{\pi}e^{-i\theta k}h(\theta)d\theta=0,\ \ k=0,1,\ldots N. \tag{11}$$

The variance of the optimal estimate of the functional is calculated by the formula

$$\left\|A_N\xi-\widehat{A}_N\xi\right\|_\alpha^\alpha=\int_{+\pi}^{\pi}\left|A_N\left(e^{i\theta}\right)-h(\theta)\right|^\alpha f(\theta)d\theta+\int_{-\pi}^{\pi}\left|h(\theta)\right|^\alpha g(\theta)d\theta. \tag{12}$$

We can conclude that the following theorem holds true.

Theorem 3.1 *Let $\{\xi_k, k\in\mathbf{Z}\}$ and $\{\eta_k, k\in\mathbf{Z}\}$ be mutually independent harmonizable symmetric α-stable random sequences which have absolutely continuous spectral measures and the spectral densities $f(\theta)>0$ and $g(\theta)>0$ satisfying the minimality condition (6). The optimal linear estimate $\widehat{A}_N\xi$ of the functional $A_N\xi=\sum_{j=0}^{N}a_j\xi_j$ that depends on the unknown values $\xi_j, j=0,1,\ldots,N$, of a random sequence $\{\xi_k, k\in\mathbf{Z}\}$ from observations of the sequence $\{\xi_k+\eta_k, k\in\mathbf{Z}\}$ at points of time $k\in\mathbf{Z}\ ,\ \ \{0,1,\ldots,N\}$ is calculated by the formula (8). The spectral characteristic $h(\theta)$ of the estimate is determined by the equation (10), where the unknown coefficients c_j are determined from the system* *of equations (11). The variance of the optimal estimate of the functional is calculated by the formula (12).*

3.1 Interpolation problem. Observation without noise

Consider the problem of optimal linear estimation of the functional $A_N\xi = \sum_{j=0}^{N} a_j \xi_j$ that depends on the unknown values of a random sequence $\{\xi_k, k \in \mathbf{Z}\}$ from observations of the sequence ξ_k at points $k \in \mathbf{Z}$, $\{0,1,\dots,N\}$. Let $\{\xi_k, k \in \mathbf{Z}\}$ be a harmonizable symmetric α -stable random sequence which have absolutely continuous spectral measure and the spectral density $f(\theta) > 0$ satisfying the minimality condition

$$\int_{-\pi}^{\pi} \left(f(\theta) \right)^{-1/(\alpha-1)} d\theta < \infty. \tag{13}$$

The optimal estimate $\widehat{A}_N\xi$ of the functional $A_N\xi$ is of the form

$$\widehat{A}_N\xi = \int_{-\pi}^{\pi} h(\theta) Z^{\xi}(d\theta). \tag{14}$$

The spectral characteristic $h(\theta)$ of the optimal linear estimate $A_N(e^{i\theta})$ of the functional is calculated by the formula

$$h(\theta) = A_N(e^{i\theta}) - \left(\overline{C_N\left(e^{i\theta}\right)} \right)^{<\frac{1}{\alpha-1}>} \left(f(\theta) \right)^{\frac{-1}{\alpha-1}}, \tag{15}$$

where the unknown coefficients c_j , $j = 0,1,\dots N$, are determined from the system of equations

$$\int_{-\pi}^{\pi} e^{-i\theta k} \left(\left(\sum_{j=0}^{N} a_j e^{ij\theta} \right) - \left(\overline{\sum_{j=0}^{N} c_j e^{ij\theta}} \right)^{<\frac{1}{\alpha-1}>} \left(f(\theta) \right)^{\frac{-1}{\alpha-1}} \right) d\theta = 0, \quad k \in \mathbf{Z} \setminus \{0,1,\dots,N\}. \tag{16}$$

The variance of the optimal estimate of the functional is calculated by the formula

$$\left\| A_N\xi - \widehat{A}_N\xi \right\|_{\alpha}^{\alpha} = \int_{+\pi}^{\pi} \left| \left(\overline{C_N\left(e^{i\theta}\right)} \right)^{<\frac{1}{\alpha-1}>} \left(f(\theta) \right)^{\frac{-1}{\alpha-1}} \right|^{\alpha} f(\theta) d\theta. \tag{17}$$

As a corollary from the theorem 3.1 the following statement holds true.

Corollary 3.1 *Let $\{\xi_k, k \in \mathbf{Z}\}$ be a harmonizable symmetric α -stable random sequence which has absolutely continuous spectral measure and the spectral density $f(\theta) > 0$ satisfying the minimality condition (13). The optimal linear estimate $\widehat{A}_N\xi$ of the functional $A_N\xi = \sum_{j=0}^{N} a_j \xi_j$, that depends on the unknown values $\xi_j, j = 0,1,\dots,N$ of the sequence $\{\xi_k, k \in \mathbf{Z}\}$ from observations of the sequence $\{\xi_k, k \in \mathbf{Z}\}$ at points of time $k \in \mathbf{Z}$, $\{0,1,\dots,N\}$ is of the form (14). The spectral characteristic $h(\theta)$ of the optimal linear estimate $\widehat{A}_N\xi$ of the functional is calculated by the formula (15), where the* *unknown coefficients* c_j, $j=0,1,\dots,N$*, are determined from the system of equations (16). The variance of the optimal estimate of the functional is calculated by the formula (17).*

3.2. Interpolation problem. Stationary sequences

Consider the problem in the particular case where $\alpha=2$. In this case the harmonizable symmetric α-stable random sequences $\{\xi_k, k\in\mathbf{Z}\}$ and $\{\eta_k, k\in\mathbf{Z}\}$ are stationary sequences and we have the problem of the optimal estimation of the linear functional $A_N\xi=\sum_{j=0}^{N}a_j\xi_j$ that depends on the unknown values ξ_j, $j=0,1,\dots,N$ of a stationary random sequence $\{\xi_k, k\in\mathbf{Z}\}$ from observations of the stationary sequence $\{\xi_k+\eta_k, k\in\mathbf{Z}\}$ at points of time $k\in\mathbf{Z}$, $\{0,1,\dots,N\}$.

The optimal linear estimate $\widehat{A}_N\xi$ of the functional is of the form (8), where the spectral characteristic $h(\theta)$ of the optimal estimate and the variance of the optimal estimate, determined by equations (10), (12), are of the form

$$h(\theta)=\frac{A_N(e^{i\theta})f(\theta)-C_N(e^{i\theta})}{f(\theta)+g(\theta)}=A_N(e^{i\theta})-\frac{A_N(e^{i\theta})g(\theta)+C_N(e^{i\theta})}{f(\theta)+g(\theta)}, \tag{18}$$

$$\left\|A_N\xi-\widehat{A}_N\xi\right\|_2^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_N(e^{i\theta})g(\theta)+C_N(e^{i\theta})\right|^2}{\left|f(\theta)+g(\theta)\right|^2}f(\theta)d\theta+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_N(e^{i\theta})f(\theta)-C_N(e^{i\theta})\right|^2}{\left|f(\theta)+g(\theta)\right|^2}g(\theta)d\theta. \tag{19}$$

The unknown coefficients c_j, $j=0,1,\dots N$, are determined from the system of equations (11) which is of the form in this case

$$\int_{-\pi}^{\pi}\left(A_N(e^{i\theta})\frac{f(\theta)}{f(\theta)+g(\theta)}-\frac{C_N(e^{i\theta})}{f(\theta)+g(\theta)}\right)e^{-ik\theta}d\theta=0, k=0,1,\dots,N.$$

From this system of equations we get the following equations

$$\sum_{j=0}^{N}a_j\int_{-\pi}^{\pi}\frac{e^{i(j-k)\theta}f(\theta)}{f(\theta)+g(\theta)}d\theta-\sum_{j=0}^{N}c_j\int_{-\pi}^{\pi}\frac{e^{i(j-k)\theta}}{f(\theta)+g(\theta)}d\theta=0,\quad k=0,1,\dots,N. \tag{20}$$

Denote by B_N, R_N, Q_N operators in the space C^{N+1} which are determined by $(N+1)\times(N+1)$ matrices with elements

$$B_{k,j}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(j-k)\theta}\frac{1}{f(\theta)+g(\theta)}d\theta,$$

$$R_{k,j}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(j-k)\theta}\frac{f(\theta)}{f(\theta)+g(\theta)}d\theta,$$

$$Q_{k,j}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(j-k)\theta}\frac{f(\theta)g(\theta)}{f(\theta)+g(\theta)}d\theta.$$

With the help of the introduced notations we can write equations (20) in the form

$$\sum_{j=0}^{N} R_{k,j}^{N} a_j = \sum_{j=0}^{N} B_{k,j}^{N} c_j, \quad k = 0,1,\ldots,N.$$

These equations can be represented in the matrix-vector form

$$R_N \mathbf{a}_N = B_N \mathbf{c}_N,$$

where $\mathbf{a}_N = (a_0, a_1, \ldots, a_N)$, $\mathbf{c}_N = (c_0, c_1, \ldots, c_N)$, and the unknown coefficients c_j, $j = 0,1,\ldots N$, can be calculated by the formula

$$c_j = \left(B_N^{-1} R_N \mathbf{a}_N\right)_j,$$

where $\left(B_N^{-1} R_N \mathbf{a}_N\right)_j$ is j-th element of the vector $\left(B_N^{-1} R_N \mathbf{a}_N\right)$.

Finally, the spectral characteristic and the variance of the optimal estimate are determined by the formulas (for more details see the books by Moklyachuk (2008), Moklyachuk and Masyutka (2012), Golichenko and Moklyachuk (2014)).

$$\begin{aligned} h(\theta) &= \frac{A_N(e^{i\theta}) f(\theta) - \sum_{j=0}^{N} \left(B_N^{-1} R_N \mathbf{a}_N\right)_j e^{ij\theta}}{f(\theta) + g(\theta)} = \\ &= A_N(e^{i\theta}) - \frac{A_N(e^{i\theta}) g(\theta) + \sum_{j=0}^{N} \left(B_N^{-1} R_N \mathbf{a}_N\right)_j e^{ij\theta}}{f(\theta) + g(\theta)}, \end{aligned} \tag{21}$$

$$\begin{aligned} \Delta(h; f; g) &= \left\| A_N\xi - \widehat{A}_N\xi \right\|_2^2 = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left| A_N(e^{i\theta}) g(\theta) + \sum_{j=0}^{N} \left(B_N^{-1} R_N \mathbf{a}_N\right)_j e^{ij\theta} \right|^2}{\left(f(\theta) + g(\theta)\right)^2} f(\theta) d\theta \\ &+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left| A_N(e^{i\theta}) f(\theta) - \sum_{j=0}^{N} \left(B_N^{-1} R_N \mathbf{a}_N\right)_j e^{ij\theta} \right|^2}{\left(f(\theta) + g(\theta)\right)^2} g(\theta) d\theta = \\ &= \left\langle R_N \mathbf{a}_N, B_N^{-1} R_N \mathbf{a}_N \right\rangle + \left\langle Q_N \mathbf{a}_N, \mathbf{a}_N \right\rangle. \end{aligned} \tag{22}$$

So, the following theorem holds true.

Theorem 3.2 *Let* $\{\xi_k, k \in \mathbf{Z}\}$ *and* $\{\eta_k, k \in \mathbf{Z}\}$ *be mutually independent harmonizable symmetric* α*-stable random sequences which have absolutely continuous spectral measures and the spectral densities* $f(\theta) > 0$ *and* $g(\theta) > 0$ *satisfying the minimality condition (6) with* $\alpha = 2$*. The optimal linear estimate* $\widehat{A}_N\xi$ *of* the linear functional $A_N\xi = \sum_{j=0}^{N} a_j \xi_j$ that *depends on the unknown values* $\xi_j, j = 0,1,\ldots,N$ *of a random sequence* $\{\xi_k, k \in \mathbf{Z}\}$ *from observations of the sequence* $\{\xi_k + \eta_k, k \in \mathbf{Z}\}$ *at* *points of time* $k \in \mathbf{Z}$, $\{0,1,\dots,N\}$*, is calculated by the formula (8). The spectral characteristic* $h(\theta)$ *of the estimate is calculated by the equation (21). The variance of the optimal estimate of the functional is calculated by the formula (22).*

3.3 Interpolation problem. Stationary sequences. Observations without noise

Consider the problem of the optimal linear estimation of the functional $A_N\xi = \sum_{j=0}^{N} a_j \xi_j$ that depends on the unknown values of a stationary random sequence $\{\xi_k, k \in \mathbf{Z}\}$ from observations of the sequence ξ_k at points $k \in \mathbf{Z}$, $\{0,1,\dots,N\}$. Let the stationary random sequence $\{\xi_k, k \in \mathbf{Z}\}$ has absolutely continuous spectral measure and the spectral density $f(\theta) > 0$ satisfying the minimality condition (13) with $\alpha = 2$. The optimal estimate $\widehat{A}_N\xi$ of the functional $A_N\xi$ is a form (18), where the spectral characteristic $h(\theta)$ of the optimal linear estimate $\widehat{A}_N\xi$ of the functional is calculated by the formula

$$h(\theta) = A_N(e^{i\theta}) - \left(\sum_{j=0}^{N} \left(B_N^{-1}\mathbf{a}_N \right)_j e^{ij\theta} \right) \left(f(\theta) \right)^{-1}. \tag{23}$$

The variance of the optimal estimate of the functional is calculated by the formula

$$\left\| A_N\xi - \widehat{A}_N\xi \right\|_2^2 = \int_{-\pi}^{\pi} \left| \sum_{j=0}^{N} \left(B_N^{-1}\mathbf{a}_N \right)_j e^{ij\theta} \right|^2 f^{-1}(\theta) d\theta = \left\langle B_N^{-1}\mathbf{a}_N, \mathbf{a}_N \right\rangle, \tag{24}$$

where B_N is an operator in the space $\square^{N+1}$, which are determined by $(N+1)\times(N+1)$ matrix with elements

$$B_N(k,j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{-1}(\theta) e^{i(j-k)\theta} d\theta, \quad k, j = 0,1,,\dots,N.$$

As a corollary from the theorem 3.2 we have the following statement.

Corollary 3.2 *Let* $\{\xi_k, k \in \mathbf{Z}\}$ *be a harmonizable symmetric* α *-stable random sequence which have absolutely continuous spectral measure and the spectral density* $f(\theta) > 0$ *satisfying the minimality condition (13) with* $\alpha = 2$*. The optimal linear estimate* $\widehat{A}_N\xi$ *of the functional* $A_N\xi = \sum_{j=0}^{N} a_j \xi_j$ *from observations of the sequence* $\{\xi_k, k \in \mathbf{Z}\}$ *at points of time* $k \in \mathbf{Z}$, $\{0,1,\dots,N\}$ *is of the form (14), where the spectral characteristic* $h(\theta)$ *of the optimal linear estimate* $\widehat{A}_N\xi$ *of the functional is calculated by the formula (23). The variance of the optimal estimate of the functional is calculated by the formula (24).*

Example 3.1 *Consider the problem of the optimal estimation of the functional* $A_0\xi = a\xi_0$ *that depends on the unknown value* ξ_0 *of a harmonizable symmetric* α *-stable random sequence* $\{\xi_k, k \in \mathbf{Z}\}$*, from observations of the sequence at points of time* $k \in \mathbf{Z} \setminus \{0\}$*. In this case the spectral characteristic* $h(\theta)$ *of the optimal estimate of the functional is of the form*

$$h(\theta)=a-c\left(f(\theta)\right)^{\frac{-1}{\alpha-1}},$$

where

$$c=a\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(\theta)\right)^{\frac{-1}{\alpha-1}}d\theta\right)^{-1}.$$

The variance of the optimal estimate of the functional is calculated by the formula

$$\left\|A_0\xi-\widehat{A}_0\xi\right\|_\alpha^\alpha=2\pi|a|^\alpha\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(\theta)\right)^{\frac{-1}{\alpha-1}}d\theta\right)^{1-\alpha}.$$

In the case of $\alpha=2$ we have the Kolmogorov (see selected works by Kolmogorov (1992)) results

$$h(\theta)=a-c\left(f(\theta)\right)^{-1},$$

$$c=a\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(\theta)\right)^{-1}d\theta\right)^{-1}.$$

$$\left\|A_0\xi-\widehat{A}_0\xi\right\|_2^2=2\pi|a|^2\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(\theta)\right)^{-1}d\theta\right)^{-1}.$$

Example 3.2 *Consider the problem of the optimal estimation of the functional $A_1\xi=\xi_0+\xi_1$ that depends on the unknown values ξ_0, ξ_1 of a harmonizable symmetric α-stable random sequence with $\alpha=4/3$ and spectral density $f(\theta)=\left|e^{i\theta}+0.5\right|^{-\frac{4}{3}}$ from observations of the sequence at points of time $n\in Z$, $\{0,1\}$. In this case the spectral characteristic $h(\theta)$ of the optimal estimate of the functional is of the form*

$$h(\theta)=1+e^{i\theta}-\left(c_0+c_1e^{-i\theta}\right)^{<3>}\left|e^{i\theta}+0.5\right|^4. \quad (25)$$

The unknown coefficients c_0 and c_1 are calculated from the indicated equations and are as follows $c_0\approx0.44,\ c_1\approx0.44$. We can represent the spectral characteristic of the optimal estimate of the functional in the form

$$h(\theta)=h_{-3}e^{-3i\theta}+h_{-2}e^{-2i\theta}+h_{-1}e^{-1i\theta}+h_2e^{2i\theta}+h_3e^{3i\theta}+h_4e^{4i\theta}$$

with

$$h_{-3}\approx-0.02, h_{-2}\approx-0.17, h_{-1}\approx-0.57, h_2\approx-0.57, h_3\approx-0.17, h_4\approx-0.02.$$

The variance of the optimal estimate of the functional is calculated by the formula

$$\left\|A_1\xi-\widehat{A}_1\xi\right\|_2^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left(0,44+0,44e^{i\theta}\right)^{<3>}\left|e^{i\theta}+0.5\right|^4\right|^{\frac{4}{3}}\left(\left|e^{i\theta}+0.5\right|^{-\frac{4}{3}}\right)d\theta\approx0,89.$$

The corresponding results for $\alpha=2$ and $f(\theta)=\left|e^{i\theta}+0.5\right|^{-2}$ are as follows:

$$c_0=\frac{4}{7},\ c_1=\frac{4}{7},$$

$$\left\|A_1\xi-\widehat{A}_1\xi\right\|_2^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left(\frac{4}{7}+\frac{4}{7}e^{i\theta}\right)^{<1>}\left|e^{i\theta}+0.5\right|^2\right|^2\left(\left|e^{i\theta}+0.5\right|^{-2}\right)d\theta=\frac{16\pi}{7}.$$

4. Interpolation Problem. The Minimax Approach

The value of the error

$$\Delta\left(h(f_0,g_0);f,g\right):=\left\|\widehat{A}_N\xi-A_N\xi\right\|_\alpha^\alpha$$

and the spectral characteristic $h(f,g):=h(\theta)$ of the optimal estimate $\widehat{A}_N\xi$ of the functional $A_N\xi$ can be calculated with the help of the proposed formulas only in the case where we know the spectral densities $f(\theta)$ and $g(\theta)$ of the harmonizable symmetric α-stable stochastic sequences $\{\xi_k,k\in\mathbf{Z}\}$ and $\{\eta_k,k\in\mathbf{Z}\}$. However, in practice we usually can't exactly evaluate the spectral densities of stochastic sequences, but, instead, we often can have a set $D=D_f\times D_g$ of admissible spectral densities. In this case we can apply the minimax-robust method of estimation to the interpolation problem. This method let us find an estimate that minimizes the maximum of the errors for all spectral densities from the given set $D=D_f\times D_g$ of admissible spectral densities simultaneously (for more details see books by Moklyachuk (2008), Moklyachuk and Masyutka (2012), Golichenko and Moklyachuk (2014)).

Definition 4.1 *For a given class of spectral densities* $D=D_f\times D_g$ *the spectral densities* $f_0(\theta)\in D_f$, $g_0(\theta)\in D_g$ *are called the least favorable in* $D=D_f\times D_g$ *for the optimal linear estimation of the functional* $A_N\xi$*, if the following relation holds true*

$$\Delta(f_0,g_0)=\Delta\left(h(f_0,g_0);f_0,g_0\right)=\max_{(f,g)\in D_f\times D_g}\Delta\left(h(f,g);f,g\right).$$

Definition 4.2 *For a given class of spectral densities* $D=D_f\times D_g$ *the spectral characteristic* $h^0=h(f_0,g_0)$ *of the optimal estimate* $\widehat{A}_N\xi$ *of the functional* $A_N\xi$ *is called minimax (robust) for the optimal linear estimation of the functional* $A_N\xi$ *if the following relations hold true*

$$h^0(\theta)\in H_D=\bigcap_{(f,g)\in D_f\times D_g}L_N^\alpha(f+g),$$

$$\min_{h\in H_D}\max_{(f,g)\in D}\Delta\left(h;f,g\right)=\max_{(f,g)\in D}\Delta\left(h^0;f,g\right).$$

The least favorable spectral densities f_0,g_0 and the minimax spectral characteristic $h^0=h(f_0,g_0)$ form a saddle point of the function $\Delta(h;f,g)$ on the set $H_D\times D$. The saddle point inequalities

$$\Delta\left(h;f_0,g_0\right)\geq\Delta\left(h^0;f_0,g_0\right)\geq\Delta\left(h^0;f,g\right)$$
$$\forall h\in H_D,\forall f\in D_f,\forall g\in D_g$$

holds true if $h^0=h(f_0,g_0)$ and $h^0(f_0,g_0)\in H_D$, where (f_0,g_0) is a solution to the conditional extremum problem

$$\sup_{(f,g)\in D_f\times D}\Delta\left(h(f_0,g_0);f,g\right)=\Delta\left(h(f_0,g_0);f_0,g_0\right),\qquad(26)$$

$$\Delta\left(h(f_0,g_0);f,g\right)=\left\|A_N\xi-\widehat{A}_N\xi\right\|_\alpha=\int_{+\pi}^{\pi}\left|A_N\left(e^{i\theta}\right)-h^0(\theta)\right|^\alpha f(\theta)d\theta+\int_{-\pi}^{\pi}\left|h^0(\theta)\right|^\alpha g(\theta)d\theta.\qquad(27)$$

The conditional extremum problem (26) is equivalent to the unconditional extremum problem

$$\Delta_D(f,g) = -\Delta\big(h(f_0,g_0);f,g\big)+\delta\big(f,g\,|\,D_f\times D_g\big)\to \inf, \tag{28}$$

where $\delta\big(f,g\,|\,D_f\times D_g\big)$ is the indicator function of the set $D_f\times D_g$. Solution (f_0,g_0) to the problem (28) is characterized by the condition $0\in\partial\Delta_D(f_0,g_0)$, where $\partial\Delta_D(f_0)$ is the subdifferential of the convex functional $\partial\Delta_D(f,g)$ at point (f_0,g_0). This condition makes it possible to find the least favorable spectral densities in some special classes of spectral densities $D=D_f\times D_g$ (Ioffe and Tihomirov, 1979; Pshenychnyj, 1971; Rockafellar, 1997; Moklyachuk, 2008b). Note, that the form (27) of the functional $\Delta\big(h(f_0,g_0);f,g\big)$ is convenient for application the Lagrange method of indefinite multipliers for finding solution to the problem (26). Making use the method of Lagrange multipliers and the form of subdifferentials of the indicator functions we describe relations that determine least favourable spectral densities in some special classes of spectral densities Summing up the derived formulas and the introduced definitions we come to conclusion that the following lemmas hold true.

Lemma 4.1 *Let the spectral densities $(f_0,g_0)\in D_f\times D_g$ give a solution to the conditional extremum problem (26). The spectral densities (f_0,g_0) are the least favorable spectral densities in $D_f\times D_g$ and $h^0=h(f_0,g_0)$ is the minimax spectral characteristic of the optimal linear estimation $\widehat{A}_N\xi$ of the functional $A_N\xi$ that depends on the unknown values ξ_j, $j=0,1,\dots,N$, of the sequence $\{\xi_k,k\in Z\}$ from observations of the sequence $\{\xi_k+\eta_k,k\in Z\}$ at points of time $k\in Z$, $\{0,1,\dots,N\}$ if $h^0=h(f_0,g_0)\in H_D$.*

Lemma 4.2 *Let the spectral density $f_0\in D_f$ gives a solution to the conditional extremum problem*

$$\sup_{f\in D_f}\Delta\big(h(f_0);f\big)=\Delta\big(h(f_0);f_0\big), \tag{29}$$

$$\Delta\big(h(f_0);f\big)=\left\|A_N\xi-\widehat{A}_N\xi\right\|_\alpha=\int_{+\pi}^{\pi}\left|\left(\overline{C_N\left(e^{i\theta}\right)}\right)^{<\frac{1}{\alpha-1}>}\left(f_0(\theta)\right)^{\frac{-1}{\alpha-1}}\right|^\alpha f(\theta)d\theta. \tag{30}$$

The spectral density f_0 is the least favorable spectral density in D_f and $h^0=h(f_0)$ is the minimax spectral characteristic of the optimal linear estimation $\widehat{A}_N\xi$ of the functional $A_N\xi$ that depends on the unknown values ξ_j, $j=0,1,\dots,N$, of the sequence $\{\xi_k,k\in Z\}$ from observations of the sequence $\{\xi_k,k\in \mathbf{Z}\}$ at points of time $k\in\mathbf{Z}$, $\{0,1,\dots,N\}$ if $h^0=h(f_0)\in H_D$.

Lemma 4.3 *Let the spectral densities $(f_0,g_0)\in D_f\times D_g$ of mutually independent stationary random sequences $\{\xi_k,k\in\mathbf{Z}\}$, $\{\eta_k,k\in\mathbf{Z}\}$ satisfy the minimality condition (6) with $\alpha=2$ and give a solution to the conditional extremum problem*

$$\sup_{(f,g)\in D_f\times D_g} \Delta\big(h(f_0,g_0);f,g\big)=\Delta\big(h(f_0,g_0);f_0,g_0\big), \tag{31}$$

$$\Delta\big(h(f_0,g_0);f,g\big)=\left\|A_N\xi-\widehat{A}_N\xi\right\|_2^2=$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_N(e^{i\theta})g_0(\theta)+\sum_{j=0}^{N}((\mathbf{B}_N^0)^{-1}\mathbf{R}_N^0\mathbf{a})_j e^{ij\theta}\right|^2}{\big(f_0(\theta)+g_0(\theta)\big)^2}f(\theta)d\theta \tag{32}$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_N(e^{i\theta})f_0(\theta)+\sum_{j=0}^{N}((\mathbf{B}_N^0)^{-1}\mathbf{R}_N^0\mathbf{a})_j e^{ij\theta}\right|^2}{\big(f_0(\theta)+g_0(\theta)\big)^2}g(\theta)d\theta.$$

The spectral densities (f_0,g_0) *are the least favorable spectral densities in* $D_f\times D_g$ *and* $h^0=(f_0,g_0)$ *is the minimax spectral characteristic of the optimal linear estimation* $\widehat{A}_N\xi$ *of the functional* $A_N\xi$ *that depends on the unknown values* $\xi_j, j=0,1,\dots,N$ *of the sequence* $\{\xi_k,k\in Z\}$ *from observations of the sequence* $\{\xi_k+\eta_k,k\in\mathbf{Z}\}$ *at points of time* $k\in\mathbf{Z}$, $\{0,1,\dots,N\}$ *if* $h^0=h(f_0,g_0)\in H_D$.

Lemma 4.4 *Let the spectral density* $f_0\in D_f$ *of a stationary random sequence* $\{\xi_k,k\in\mathbf{Z}\}$ *satisfies the minimality condition (13) with* $\alpha=2$ *and gives a solution to the conditional extremum problem*

$$\sup_{f\in D_f}\Delta\big(h(f_0);f\big)=\Delta\big(h(f_0);f_0\big), \tag{33}$$

$$\Delta\big(h(f_0);f\big)=\left\|A_N\xi-\widehat{A}_N\xi\right\|_2^2=\int_{-\pi}^{\pi}\left|\sum_{j=0}^{N}\big((\mathbf{B}_N^0)^{-1}\mathbf{a}_N\big)_j e^{ij\theta}\right|^2 f_0^{-2}(\theta)f(\theta)d\theta. \tag{34}$$

The spectral density f_0 *is the least favorable spectral density in* D_f *and* $h^0=h(f_0,g_0)$ *is the minimax spectral characteristic of the optimal linear estimation* $\widehat{A}_N\xi$ *of the functional* $A_N\xi$ *that depends on the unknown values* $\xi_j, j=0,1,\dots,N$, *of the sequence* $\{\xi_k,k\in Z\}$ *from observations of the sequence* $\{\xi_k,k\in\mathbf{Z}\}$ *at points of time* $k\in\mathbf{Z}\setminus\{0,1,\dots,N\}$ *if* $h^0=h(f_0)\in H_D$.

5. Least Favourable Spectral Densities In The Class $D_f^{\beta}\times D_g^{v,u}$

Consider the problem of optimal estimation of the linear functional $A_N\xi$ that depends on the unknown values $\xi_j, j=0,1,\dots,N$, of a random sequence $\{\xi_k,k\in\mathbf{Z}\}$ from observations of the sequence $\{\xi_k+\eta_k,k\in\mathbf{Z}\}$ at points of time $k\in\mathbf{Z}$, $\{0,1,\dots,N\}$, where $\{\xi_k,k\in\mathbf{Z}\}$ and $\{\eta_k,k\in\mathbf{Z}\}$ are mutually independent harmonizable symmetric α-stable random sequences which have spectral densities $f(\theta)>0$ and $g(\theta)>0$ satisfying the minimality condition (6) from the class of admissible spectral densities $D=D_f^{\beta}\times D_g^{v,u}$, where

$$D_f^{\beta}=\left\{f(\theta)\left|\int_{-\pi}^{\pi}\left(f(\theta)\right)^{\beta}d\theta=P_1\right.\right\},$$

$$D_g^{v,u}=\left\{g(\theta)\left|v(\theta)\le g(\theta)\le u(\theta),\int_{-\pi}^{\pi}g(\theta)d\theta=P_2\right.\right\}.$$

Assume that the spectral densities $f_0\in D_f^{\beta}$, $g_0\in D_g^{v,u}$ and the functions $h_f(f_0,g_0)$, $h_g(f_0,g_0)$ determined by the equations

$$h_f(f_0,g_0)=\left|A_N(e^{i\theta})-h^0(\theta)\right|^{\alpha}, \tag{35}$$

$$h_g(f_0,g_0)=\left|h^0(\theta)\right|^{\alpha}, \tag{36}$$

$$\left(A_N\left(e^{i\theta}\right)-h^0(\theta)\right)^{<\alpha-1>}f_0(\theta)-\left(h^0(\theta)\right)^{<\alpha-1>}g_0(\theta)=\overline{C_N\left(e^{i\theta}\right)}, \tag{37}$$

$$\int_{-\pi}^{\pi}e^{-i\theta k}h^0(\theta)d\theta=0,k=0,1,\dots N. \tag{38}$$

are bounded. Under these conditions the functional

$$\Delta\left(h(f_0,g_0);f,g\right)=\left\|A_N\xi-\widehat{A}_N\xi\right\|_{\alpha}=\int_{+\pi}^{\pi}\left|A_N\left(e^{i\theta}\right)-h^0(\theta)\right|^{\alpha}f(\theta)d\theta-\int_{-\pi}^{\pi}\left|h^0(\theta)\right|^{\alpha}g(\theta)d\theta \tag{39}$$

is linear and continuous in the $L_1\times L_1$ space and we can apply the Lagrange multipliers method to derive that the least favorable densities $f_0\in D_f^{\beta}$, $g_0\in D_g^{v,u}$ satisfy the equations

$$\left|A_N\left(e^{i\theta}\right)-h^0(\theta)\right|^{\alpha}=\gamma_1\left(f_0(\theta)\right)^{\beta-1}, \tag{40}$$

$$\left|h^0(\theta)\right|^{\alpha}=\gamma_2\left(\phi_1(\theta)+\phi_2(\theta)+\gamma_3\right), \tag{41}$$

where $\phi_1(\theta)\le 0$ and $\phi_1(\theta)=0$ if $g_0(\theta)\ge v(\theta)$; $\phi_2(\theta)\ge 0$ and $\phi_2(\theta)=0$ if $g_0(\theta)\le u(\theta)$; $\gamma_1,\gamma_2,\gamma_3$ are the Lagrange multipliers which are determined from the conditions

$$\int_{-\pi}^{\pi}\left(f_0(\theta)\right)^{\beta}d\theta=P_1,\ \int_{-\pi}^{\pi}g_0(\theta)d\theta=P_2.$$

Thus, the following statement holds true.

Theorem 5.1 *Let the spectral densities $f_0\in D_f^{\beta}$, $g_0\in D_g^{v,u}$ satisfy the minimality condition (6) and let the functions $h_f(f_0,g_0)$, $h_g(f_0,g_0)$ determined by formulas (35), (36), (37), (38) be bounded. The spectral densities f_0 and g_0 are the least favorable in the class $D=D_f^{\beta}\times D_g^{v,u}$ for the optimal linear estimation of the functional $A_N\xi$ if they satisfy equations (40), (41) and determine a solution to the* *extremum problem (26). The minimax-robust spectral characteristic $h\left(f_0, g_0\right)$ of the optimal estimate of the functional $A_N\xi$ is determined by formulas (37), (38).*

5.1 Least favorable spectral densities. Observations without noise

Consider the problem of optimal linear estimation of the functional $A_N\xi$ that depends on the unknown values of a sequence $\xi_k, k\in\mathbb{Z}$, from observations of the sequence $\{\xi_k, k\in\mathbb{Z}\}$ at points $k\in\mathbb{Z}$, $\{0,1,\dots,N\}$, where $\{\xi_k, k\in\mathbb{Z}\}$ is a harmonizable symmetric α-stable random sequence which have spectral density $f(\theta)>0$ satisfying the minimality condition (13) from the class of admissible spectral densities D_f^β. Assume that spectral density $f_0\in D_f^\beta$ and the function $h_f(f_0, g_0)$ determined by the equation

$$h_f(f_0) = \left|\left(\overline{C_N^0(e^{i\theta})}\right)^{<\frac{1}{\alpha-1}>}(f_0(\theta))^{\frac{-1}{\alpha-1}}\right|^\alpha \tag{42}$$

is bounded.
Under this condition the functional (34) is linear and continuous in the L_1 space and we can apply the Lagrange multipliers method to find solution of the conditional extremum problem (29) and derive that the least favorable density $f^0\in D_f^\beta$ satisfy the equation

$$\left|\left(\overline{C_N^0(e^{i\theta})}\right)^{<\frac{1}{\alpha-1}>}(f_0(\theta))^{\frac{-1}{\alpha-1}}\right|^\alpha = \gamma_1\left(f_0(\theta)\right)^{\beta-1}, \tag{43}$$

where γ_1 is the Lagrange multipliers. From this equation we find that the least favorable density is of the form

$$f_0(\theta) = C\left|\sum_{j=0}^{N}\overline{c_j}e^{-ij\theta}\right|^{\frac{\alpha}{\alpha+(\alpha-1)(\beta-1)}}. \tag{44}$$

The unknown constants are determined from the extremum problem (29) and from the condition

$$\int_{-\pi}^{\pi}(f_0(\theta))^\beta d\theta = P_1.$$

In the case $\beta=1$ the least favorable density is of the form

$$f_0(\theta) = C\left|\sum_{j=0}^{N}\overline{c_j}e^{-ij\theta}\right|. \tag{45}$$

The following statement holds true.

Theorem 5.2 *Let the spectral density $f_0\in D_f^\beta$ satisfy the minimality condition (13) and let the function $h_f(f_0)$ determined by formula (42) be bounded. The spectral density $f_0(\theta)$ is the least favorable in the class D_f^β for the optimal linear estimation of the functional $A_N\xi$ if it is of the form*

(44) and determines a solution to the extremum problem (29). The minimax-robust spectral characteristic $h(f_0)$ of the optimal estimate of the functional $A_N\xi$ is determined by formulas (15).

5.2 Least favorable spectral densities. Stationary sequences

Consider the problem of the optimal estimation of the linear functional $A_N\xi$ that depends on the unknown values $\xi_j, j=0,1,\dots,N$, of a random sequence $\{\xi_k, k\in\mathbb{Z}\}$ from observations of the sequence $\{\xi_k+\eta_k, k\in\mathbb{Z}\}$ at points of time $k\in\mathbb{Z}$, $\{0,1,\dots,N\}$, where $\{\xi_k, k\in\mathbb{Z}\}$ and $\{\eta_k, k\in\mathbb{Z}\}$ are mutually independent stationary random sequences which have spectral densities $f(\theta)>0$ and $g(\theta)>0$ satisfying the minimality condition (13) with $\alpha=2$ from the class of admissible spectral densities $D=D_f^{\beta}\times D_g^{v,u}$.

Assume that spectral densities $f_0\in D_f^{\beta}$, $g_0\in D_g^{v,u}$ and the functions $h_f(f_0,g_0)$, $h_g(f_0,g_0)$, determined by the equations

$$h_f(f_0,g_0)=\frac{\left|A_N(e^{i\theta})g_0(\theta)+\sum_{j=0}^{N}((\mathbf{B}_N^0)^{-1}\mathbf{R}_N^0\mathbf{a})_j e^{ij\theta}\right|^2}{(f_0(\theta)+g_0(\theta))^2}, \tag{46}$$

$$h_g(f_0,g_0)=\frac{\left|A_N(e^{i\theta})f_0(\theta)-\sum_{j=0}^{N}((\mathbf{B}_N^0)^{-1}\mathbf{R}_N^0\mathbf{a})_j e^{ij\theta}\right|^2}{(f_0(\theta)+g_0(\theta))^2} \tag{47}$$

are bounded. Under these conditions the functional (32) is linear and continuous in the $L_1\times L_1$ space and we can apply the Lagrange multipliers method to find solution of the conditional extremum problem (31) and derive that the least favorable densities $f^0\in D_f^{\beta}$, $g^0\in D_g^{v,u}$ satisfy the equations

$$\left|A_N(e^{i\theta})g_0(\theta)+\sum_{j=0}^{N}((\mathbf{B}_N^0)^{-1}\mathbf{R}_N^0\mathbf{a})_j e^{ij\theta}\right|^2=\gamma_1(f_0(\theta)+g_0(\theta))^2\left(f_0(\theta)\right)^{\beta-1}, \tag{48}$$

$$\left|A_N(e^{i\theta})f_0(\theta)-\sum_{j=0}^{N}((\mathbf{B}_N^0)^{-1}\mathbf{R}_N^0\mathbf{a})_j e^{ij\theta}\right|^2=\gamma_2(f_0(\theta)+g_0(\theta))^2\left(\varphi_1(\theta)+\varphi_2(\theta)+\gamma_3\right), \tag{49}$$

where $\varphi_1(\theta)\le 0$ and $\varphi_1(\theta)=0$ if $g_0(\theta)\ge v(\theta)$; $\varphi_2(\theta)\ge 0$ and $\varphi_2(\theta)=0$ if $g_0(\theta)\le u(\theta)$; γ_1, γ_2, γ_3 are the Lagrange multipliers which are determined from the conditions

$$\int_{-\pi}^{\pi}(f_0(\theta))^{\beta}d\theta=P_1,\quad \int_{-\pi}^{\pi}g_0(\theta)d\theta=P_2.$$

Thus, the following statement holds true.

Theorem 5.3 *Let the spectral densities $f_0\in D_f^{\beta}$, $g_0\in D_g^{\beta}$ satisfy the minimality condition (6) with $\alpha=2$ and let the functions $h_f(f_0,g_0)$, $h_g(f_0,g_0)$ determined by formulas (46), (47) be bounded.*

Spectral densities $f_0(\theta)$ and $g_0(\theta)$ are the least favorable in the class $D = D_f^\beta \times D_g^{v,u}$ for the optimal linear estimation of the functional $A_N\xi$ if they satisfy equations (48), (49) and determine a solution to the extremum problem (31). The minimax-robust spectral characteristic $h(f_0, g_0)$ of the optimal estimate of the functional $A_N\xi$ is determined by formulas (21).

5.3 Least favorable spectral densities. Stationary sequences. Observations without noise

Consider the problem of the optimal linear estimation of the functional $A_N\xi$ that depends on the unknown values of a random sequence $\{\xi_k, k \in \mathbb{Z}\}$, from observations of the sequence $\{\xi_k, k \in \mathbb{Z}\}$, at points $k \in \mathbb{Z}$, $\{0,1,\ldots,N\}$, where the stationary random sequence $\{\xi_k, k \in \mathbb{Z}\}$, has the spectral density $f(\theta) > 0$ satisfying the minimality condition (13) with $\alpha = 2$ from the class of admissible spectral densities D_f^β. Assume that spectral density $f_0 \in D_f^\beta$ and the function $h_f(f_0)$ determined by the equation

$$h_f(f_0) = \left|\sum_{j=0}^{N} ((\mathbf{B}_N^0)^{-1}\mathbf{a})_j e^{ij\theta}\right|^2 f_0^{-2}(\theta) \tag{50}$$

is bounded.

Under this condition the functional (34) is linear and continuous in the L_1 space and we can apply the Lagrange multipliers method to find solution of the conditional extremum problem (33) and derive that the least favorable density $f^0 \in D_f^\beta$ satisfy the equation

$$\left|\sum_{j=0}^{N} ((\mathbf{B}_N^0)^{-1}\mathbf{a})_j e^{ij\theta}\right|^2 f_0^{-2}(\theta) = \gamma_1 \left(f_0(\theta)\right)^{\beta-1} \tag{51}$$

where γ_1 is the Lagrange multiplier which is determined from the conditions

$$\int_{-\pi}^{\pi} (f_0(\theta))^\beta d\theta = P_1.$$

Thus, the following statement holds true.

Theorem 5.4 *Let the spectral density $f_0 \in D_f^\beta$ satisfy the minimality condition (13) with $\alpha = 2$ and let the function $h_f(f_0)$ determined by formula (50) be bounded. The spectral density $f_0(\theta)$ is the least favorable in the class D_f^β for the optimal linear estimation of the functional $A_N\xi$ if it satisfies equation (51) and determine a solution to the extremum problem (33). The minimax-robust spectral characteristic $h(f_0, g_0)$ of the optimal estimate of the functional $A_N\xi$ is determined by formulas (23).*

5.4 Least favorable spectral densities in the class D_f^{-1}

Consider the problem of the optimal estimation of the functional $A_N\xi = \sum_{j=0}^{N} a_j\xi_j$ which depends on the unknown values of a stationary stochastic sequence ξ_j in the case where the spectral density is from the class D_f^{β}, where $\beta=-1$, and the sequence $a_j, j=0,1,\dots,N$, that determines the functional $A_N\xi$, is strictly positive. By using the method of Lagrange multipliers we get that the Fourier coefficients of the function f_0^{-1} satisfy the equation

$$\left|\sum_{j=0}^{N}\overline{c_j}e^{-ij\theta}\right|^2 = p_0^2, \tag{52}$$

where $c_j, j=0,1,\dots,N$, are components of the vector $\mathbf{c}_N$ that satisfies the equation $\mathbf{B}_N^0\mathbf{c}_N=\mathbf{a}_N$, the matrix $\mathbf{B}_N^0$ is determined by the Fourier coefficients of the function $f_0^{-1}(\theta)$

$$B_N^0(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f_0^{-1}(\theta)e^{-i(k-j)\theta}d\theta=r_{k-j}^0,$$
$$k,j=0,1,\dots,N.$$

The Fourier coefficients $r_k=r_{-k}, k=0,1,\dots,N$, satisfy both equation (52) and equation $\mathbf{B}_N^0\mathbf{c}_N=\mathbf{a}_N$. These coefficients can be found from the equation $\mathbf{B}_N^0\mathbf{p}_N^0=\mathbf{a}_N$, where $\mathbf{p}_N^0=(p_0,0,\dots,0)$. The last relation can be presented in the form of the system of equations

$$r_k p_0 = a_k, k=0,1,\dots,N.$$

From the first equation of the system (for $k=0$) we find the unknown value $p_0=a_0(r_0)^{-1}$. It follows from the restriction on the spectral densities from the class D_0^{-1} that the Fourier coefficient

$$r_0=\int_{-\pi}^{\pi} f_0^{-1}(\theta)d\theta=P_1.$$

For this reason $r_k=r_{-k}=P_1a_ka_0^{-1}, k=1,\dots$. We can represent the function $f_0^{-1}(\theta)$ in the form $f_0^{-1}(\theta)=\sum_{k=-N}^{N} r_ke^{ik\theta}$. Since the sequence $a_j, j=0,1,\dots,N$ is strictly positive, the sequence $r_k, k=0,1,\dots,N$ is also strictly positive and the function $f_0^{-1}(\theta)$ is positive, so it can be represented in the form (Hannan, 1970; Krein and Nudelman, 1977)

$$f_0^{-1}(\theta)=\left|\sum_{k=0}^{N}\gamma_ke^{-ik\theta}\right|^2, \theta\in[-\pi,\pi].$$

Hence, $f_0(\theta)$ is the spectral density of the autoregressive stochastic sequence of the order N generated by equation

$$\sum_{k=0}^{N}\gamma_k\xi(n-k)=\varepsilon_n, \tag{53}$$

where ε_n is a ``white noise'' sequence. Thus, the following theorem holds true.

Theorem 5.5 The least favorable in the class D_0^{-1} spectral density for the optimal linear estimation of the functional $A_N\xi$ determined by strictly positive sequence $a_j, j=0,1,\ldots,N$, is the spectral density of the autoregressive sequence (53) whose Fourier coefficients are $r_k = r_{-k} = P_1 a_k a_0^{-1}, k=0,1,\ldots,N.$

6. Conclusion

In this article we propose methods of solution of the optimal linear estimation problem for the linear functional $A_N\xi = \sum_{i=0}^{N} a_i\xi_i$ that depends on the unknown values of a random sequence $\{\xi_k, k\in\mathbb{Z}\}$, from observations of the sequence $\{\xi_k+\eta_k, k\in\mathbb{Z}\}$ at points of time $k\in\mathbb{Z}$, $\{0,1,\ldots,N\}$, where $\{\xi_k, k\in\mathbb{Z}\}$ and $\{\eta_k, k\in\mathbb{Z}\}$ are mutually independent harmonizable symmetric α-stable random sequences which have the spectral densities $f(\theta)>0$ and $g(\theta)>0$ satisfying the minimality condition. The problem is investigated under the condition of spectral certainty as well as under the condition of spectral uncertainty. Formulas for calculation the value of the error and the spectral characteristic of the optimal linear estimate of the functional are derived under the condition of spectral certainty where spectral densities of sequences are exactly known. In the case where spectral densities of sequences are not exactly known, but classes of admissible spectral densities are available, relations which determine the least favorable densities and the minimax-robust spectral characteristics for different classes of spectral densities are found.

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