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Research Article

Justification of Application of the Fourier Method for Parabolic Equations with Random Boundary Conditions from Orlicz Spaces

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Abstract

A new method of constructing solutions of boundary value problems for parabolic equations with random boundary value conditions is proposed. We assume that the boundary conditions are stochastic processes from the Orlicz space of random variables (including processes with zero mean values).

Keywords: Mathematical physics; Parabolic equations; Stochastic processes; Boundary value problem; Accuracy and Reliability; Fourier method; Convergence in Orlicz spaces; Distribution of supremum

Subject classification: 60G60; Secondary 60G15

1. Introduction

Investigation of solutions of equations of mathematical physics with random factors is an interesting and important problem of probability theory. These equations were studied by Buldygin and Kozachenko (2000); Dovgaj et al. (2008). Buldygin and Kozachenko (2000) proposed a new method of investigation of problems of mathematical physics, which allows application of the Fourier method for finding solutions for these problems and studying properties of solutions.

They studied the first boundary value problem for homogeneous hyperbolic equation with random Gaussian initial conditions. In the paper by Kozachenko and Barrasa de la Krus (1995) the first boundary value problem was studied in the case where the initial conditions are random processes from the Orlicz spaces. In the multidimensional case the homogeneous hyperbolic equations with random sub-Gaussian initial conditions are investigated in the book by Dovgaj et al. (2008). They

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investigated the boundary value problem for nonhomogeneous equations with random right-hand sides which are φ-sub-Gaussian random fields.

In this paper we propose a new method of construction solutions of the boundary-value problems for parabolic equations with random boundary conditions. Similar problems for hyperbolic and parabolic equations with random initial conditions were investigated by Kozachenko and Koval'chuk (1998); Barrasa de la Krus and Kozachenko (1995); Kozachenko and Veresh (2010); Slyvka-Tylyshchak and Veresh (2008); Slyvka Tylyshchak (2012).

A survey of the corresponding results can be found in books by Buldygin and Kozachenko (2000), Dovgaj et al. (2008).

We analyze conditions justifying application of the Fourier method for parabolic equations with random boundary conditions and obtain bounds for the distribution of supremum of solutions of these equations. We assume that the boundary conditions are stochastic processes belonging to the Orlicz space of random variables (including processes with zero mean values).

Results presented in this paper have theoretical and practical application in studying the parabolic equations of mathematical physics with random boundary conditions. Moreover, these results allow to model solutions of these equations.

The paper has the following structure. Section 2 contains the necessary definitions and results from the theory of Orlicz spaces. The description of the problem as well as statements of the main results of the paper are given in Section 3.

2. Stochastic processes from the Orlicz Space

In this section we present the necessary definitions and results of the theory of Orlicz spaces.

Definition 2.1 (Buldygin and Kozachenko, 2000). *A continuous even convex function $U(x), x \in R$, is called C-function if $U(x)$ is monotone increasing function for $x > 0$ and $U(0) = 0$.*

Definition 2.2 (Buldygin and Kozachenko, 2000). *We say that C-function $U = \{U(x), x \in R\}$ satisfies g-condition if there exist constants $z_0 \geq 0, K > 0$ and $A > 0$, such that the inequality*

$$U(x)U(y) \leq AU(Kxy)$$

holds true for all $x \geq z_0$ and all $y \geq z_0$.

Definition 2.3 (Buldygin and Kozachenko, 2000). *The space $L_U(\Omega)$ of random variables $\xi = \xi(\omega), \omega \in \Omega$, is called the Orlicz space generating by C-function $U(x)$ if for any $\xi \in L_U(\Omega)$ there exist a constant $r_\xi > 0$, such that*

$$EU\left(\frac{\xi}{r_\xi}\right) < \infty.$$

The Orlicz space $L_U(\Omega)$ is a Banach space with respect to the norm

$$\|\xi\|_{L_U} = inf\left\{r > 0: EU\left(\frac{\xi}{r_\xi}\right) \leq 1\right\}.$$

Definition 2.4 (Buldygin and Kozachenko, 2000). *Let $X = \{X(t), t \in T\}$ be a stochastic process. We say that X belongs to the Orlicz space $L_U(\Omega)$ if for each $t \in T$ the random variable $X(t)$ belongs to the space $L_U(\Omega)$.*

Definition 2.5 (Barrasa de la Krus and Kozachenko, 1995). *Let $U(x)$ be a C-function. A family Δ of the centered random variables ξ, $E\xi = 0$, from the Orlicz space $L_U(\Omega)$ is called strictly Orlicz family if there exist a constant C_Δ such that*

$$\left\|\sum_{i\in I}\lambda_i\xi_i\right\|_{L_U} \le C_\Delta\left(E\left(\sum_{i\in I}\lambda_i\xi_i\right)^2\right)^{\frac{1}{2}}.$$

for all finite collections of random variables $\xi_i \in \Delta$, $i \in I$, and for all $\lambda_i \in R^1$.

Definition 2.6 (Barrasa de la Krus and Kozachenko, 1995). *A stochastic process $X = \{X(t), t \in T\}$, $(X(t) \in L_U(\Omega))$, is called strictly Orlicz process, if the collection of the random variables $\{X(t), t \in T\}$ is a strictly Orlicz family. Two stochastic processes $X = \{X(t), t \in T\}$ and $Y = \{Y(t), t \in T\}$ are called jointly strictly Orlicz processes if the collection of the random variables $\{X(t), Y(t), t \in T\}$ is a strictly Orlicz family.*

Theorem 2.1 (Dovgaj et al., 2008). *Let $X_i = \{X_i(t), t \in T\}$, $i \in I$, be a family of jointly strictly Orlicz processes. If there exist the integral in the mean square sense*

$$\xi_{ki} = \int_T \varphi_k(t) X_i(t) d\mu(t),$$

then the family of random variables $\Delta_\xi = \{\xi_{ki}, i \in I, k \in N\}$ is a strictly Orlicz family.

The next theorem is a particular case of the theorem proved by Kozachenko and Veresh (2010).

Theorem 2.2 *Let $X_n = \{X_n(t), t \in T\}$, $n = 1, 2, \ldots$, where $T \subset R^2$, $T = \{0 \le t_i \le T_i, i = 1, 2\}$,* $d(t,s) = \max_{i=1,2}|t_i - s_i|$, be a *sequence of stochastic processes belonging to the Orlicz space, and let the function U satisfies the g −condition. Let the following conditions hold true:*

1. *$X_n(t)$ are separable processes;*
2. *$X_n(t) \to X(t)$ as $n \to \infty$, $t \in T$, in probability;*
3. $\sup_{d(t,s)\le h}\sup_{n=\overline{1,\infty}}\|X_n(t) - X_n(s)\| \le \sigma(h)$, *where $\sigma = \{\sigma(h), h \ge 0\}$ is a continuous increasing function, such that $\sigma(h) \to 0$ as $h \to 0$;*
4. *for some $\varepsilon > 0$*

$$\int_0^\varepsilon U^{(-1)}\left(\left(\frac{T_1}{2\sigma^{(-1)}(u)} + 1\right)\left(\frac{T_2}{2\sigma^{(-1)}(u)} + 1\right)\right) du < \infty,$$

where $\sigma^{(-1)}(u)$ is the inverse function to $\sigma(u)$.
Then the sequence of processes $X_n(t)$ converges in probability to $X(t)$ in the space $C(T)$.

Lemma 2.1 (Kozachenko and Veresh, 2010). *Let $Y_\lambda(u)$, $\lambda > 0$, $u \in T$, $T \in (0, \infty)$, be a function such that:*

1. $\sup_{u\in T} |Y_\lambda(u)| \le B,$
2. $|Y_\lambda(u) - Y_\lambda(v)| \le C\lambda\,|u - v|$ *for all* $u, v \in T.$

Let $\varphi(\lambda), \lambda > 0$, *be a continuous increasing function,* $\varphi(\lambda) > 0$ *for all* $\lambda > 0$, *such that the function* $\frac{\lambda}{\varphi(\lambda)}$ *is increasing for* $\lambda > v_0$ *with a constant* $v_0 \ge 0$. *Then for all* $\lambda \ge 0$ *and* $v > 0$ *the following inequality holds true*

$$|Y_\lambda(u) - Y_\lambda(v)| \le max(C, 2B)\,\frac{\varphi(\lambda + v_0)}{\varphi\left(\frac{1}{|u-v|} + v_0\right)}.$$

Definition 2.7 (Darijchuk et al., 2011) *A monotone nondecreasing sequence of positive numbers* $(\chi_U(n), n \ge 1)$ *is called M-characteristic (majorant characteristic) of the space* $L_U(\Omega)$, *if for any* $n \in N$, *and* $\xi_k \in L_U(\Omega), k = 1, \dots, n$, *the following inequality holds:*

$$\left\| \max_{1\le k\le n} |\xi_k| \right\|_{L_U} \le \chi_U(n) \max_{1\le k\le n} \|\xi_k\|_{L_U}.$$

Theorem 2.3 (Darijchuk et al., 2011) *Let* (T, ρ) *be a metric compact space, let* $N(u)$ *be the metric massiveness of the space* (T, ρ), *let* $X = \{X(t), t \in T\}$ *be a separable stochastic process belonging to the space* $L_U(\Omega)$, *let* $\chi_U(n)$ *be the M-characteristic (majorant characteristic) of the space* $L_U(\Omega)$. *Let there exist a function*

$$\sigma = \left\{ \sigma(h), 0 \le h \le \sup_{t,s\in T} \rho(t,s) \right\},$$

where $\sigma(h)$ *is a continuous monotone increasing function,* $\sigma(0) = 0$, *and let*

$$\sup_{\rho(t,s)\le h} \|X(t) - X(s)\|_{L_U} \le \sigma(h).$$

If for some ε

$$\int_0^{\varepsilon} \chi_U\left(N\left(\sigma^{(-1)}(u)\right)\right) du < \infty,$$

where $\sigma^{(-1)}(u)$ *is the inverse function to* $\sigma(h)$, *then* $\sup_{t\in T} X(t)$ *with probability one belongs to the space* $L_U(\Omega)$ *and the following inequality holds true*

$$\sup_{t\in T} |X(t)|_{L_U} \le \|X(t_0)\|_{L_U} + \frac{1}{\theta(1-\theta)} \int_0^{\omega_0\theta} \chi\left(N\left(\sigma^{(-1)}(u)\right)\right) du = B(t_0, \theta),$$

where t_0 *is any point from* T, $\omega_0 = \sigma\left(\sup_{t\in T} \rho(t_0, t)\right), 0 < \theta < 1.$

Moreover, for any $\varepsilon > 0$ *the following inequality holds true*

$$P\left\{ \sup_{t\in T} |X(t)| > \varepsilon \right\} \le \left(U\left(\frac{\varepsilon}{B(t_0,\theta)} \right) \right)^{-1}.$$

Corollary 2.1 (Darijchuk et al., 2011) *Let in Theorem 2.3* $T = \{0 \le x \le b, c \le t \le d\}$, $\rho((x,t),(x_1,t_1)) = max\,(|x - x_1|, |t - t_1|)$. *Then condition*

$$\int_0^{\varepsilon} \chi_U\left(N\left(\sigma^{(-1)}(u)\right)\right) du < \infty$$

holds if for some $\varepsilon > 0$

$$\int_0^{\varepsilon} U^{(-1)}\left(\left(\frac{b}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{d-c}{2\sigma^{(-1)}(u)}+1\right)\right) du < \infty,$$

and

$$B(t_0,\theta) \le \hat{B}(t_0,\theta) = \|X(t_0)\|_{L_U} + \frac{1}{\theta(1-\theta)}\int_0^{\omega_0\theta} \chi_U\left(\left(\frac{b}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{d-c}{2\sigma^{(-1)}(u)}+1\right)\right) du,$$

and for any $\varepsilon > 0$

$$P\left\{\sup_{t\in T}|X(t)| > \varepsilon\right\} \le \left(U\left(\frac{\varepsilon}{\hat{B}(t_0,\theta)}\right)\right)^{-1}.$$

3. Formulation the Problem and Main Results

Consider the boundary value problem for a parabolic equation with two independent variables ($x \in [0,l], t \in [0,T], T > 0$), namely:

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial z(t,x)}{\partial x}\right) - q(x)z(t,x) - \rho(x)\frac{\partial z(t,x)}{\partial t} = 0, 0 < x < l, 0 < t < T < \infty, \tag{3.1}$$

$$z(0, x) = 0, \ 0 \le x \le l, \tag{3.2}$$

$$z_x(t,0) - \alpha z(t,0) = \eta_1(t), \ z_x(t,l) + \alpha z(t,l) = \eta_2(t), \ 0 \le t < T < \infty, \alpha \in R, \tag{3.3}$$

where $\eta_1(t), \eta_2(t), 0 \le t < T < \infty$ are independent stochastic processes from the Orlicz space $L_U(\Omega)$. We will suppose that stochastic processes have zero mean values $E\eta_1(t) = 0$, $E\eta_2(t) = 0$, and the covariance functions $E\eta_1(t)\eta_1(s) = B_1(t,s)$, $E\eta_2(t)\eta_2(s) = B_2(t,s)$ of the processes are continuous.

Remark 3.1 If we replace conditions (3.3) by the conditions

$$\gamma z_x(t,0) - \beta z(t,0) = \gamma\eta_1(t), \gamma z_x(t,l) + \beta z(t,l) = \gamma\eta_2(t), 0 \le t < T < \infty, \beta, \ \gamma \in R, \tag{3.4}$$

and take $\gamma = 0, \beta \in R$, we get the problem investigated in the paper by Kozachenko and Veresh (2010). In the case $\gamma \in R\setminus\{0\}, \beta \in R$, conditions (3.4) cover all known boundary value conditions except boundary value conditions from the paper by Kozachenko and Veresh (2010). The following results are generalization of results described in the paper by Kozachenko and Veresh (2010).

Let the functions $p(x)$, $q(x)$, $\rho(x)$ satisfy the following conditions:
1. $p(x)$, $p'(x)$, $q(x)$, $\rho(x)$ are continuous functions on the interval $[0,l]$;
2. $p(x) > 0, \rho(x) > 0, \forall x \in [0,l]$.

The problem (3.1)-(3.3) can be reduced to the following problem (for reducing we take $\eta_1(0)=\eta_2(0)=0$)

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial W(t,x)}{\partial x}\right)-q(x)W(t,x)-\rho(x)\frac{\partial W(t,x)}{\partial t}=-\rho(x)\gamma(t,x), 0<x<l, 0<t<T<\infty, \quad (3.5)$$

$$W(0,x)=0, 0\le x\le l, \quad (3.6)$$

$$W_x(t,0)-\alpha W(t,0)=0, W_x(t,l)+\alpha W(t,l)=0, 0\le t<T<\infty, \alpha\in R, \quad (3.7)$$

where

$$W(t,x)=z(t,x)-\left(\frac{1}{2l+\alpha l^2}\eta_2(t)-\frac{1+\alpha l}{2l+\alpha l^2}\eta_1(t)\right)x^2-\eta_1(t)x,$$

$$-\rho(x)\gamma(t,x)=\frac{q(x)x^2-2p'(x)x-2p(x)}{2l+\alpha l^2}\eta_2(t)+$$

$$+\frac{1+\alpha l}{2l+\alpha l^2}\left(-q(x)x^2+2p'(x)x+2p(x)+\frac{(q(x)x-p'(x))(2l+\alpha l^2)}{1+\alpha l}\right)\eta_1(t)+$$

$$+\frac{\rho(x)x^2}{2l+\alpha l^2}\eta_2'(t)+\left(\rho(x)x-\frac{\rho(x)x^2(1+\alpha l)}{2l+\alpha l^2}\right)\eta_1'(t).$$

Solution of the problem (3.5)-(3.7) we will seek in the following form

$$W(t,x)=\sum_{n=1}^{\infty}X_n(x)\int_0^t\gamma_n(t')\,e^{-\lambda_n(t-t')}dt'. \quad (3.8)$$

In this case solution of the problem (3.1)-(3.3) has the form

$$z(t,x)=\left(\frac{1}{2l+\alpha l^2}\eta_2(t)-\frac{1+\alpha l}{2l+\alpha l^2}\eta_1(t)\right)x^2+\eta_1(t)x+\sum_{n=1}^{\infty}X_n(x)\int_0^t\gamma_n(t')\,e^{-\lambda_n(t-t')}dt', \quad (3.9)$$

where $\gamma_n(t)$ are the Fourier coefficients of the function $\gamma(t,x)$ as a function of x, namely

$$\gamma_n(t)=\int_0^t\rho(x)\gamma(t,x)X_n(x)dx.$$

Here $X_n(x)$ are eigenfunctions corresponding to eigenvalues λ_n, $\lambda_n>0$, of the Sturm-Liouville problem

$$\frac{d}{dx}\left(p(x)\frac{dX(x)}{dx}\right)-q(x)X(x)+\lambda\rho(x)X(x)=0, \quad (3.10)$$

$$X'(0)-\alpha X(0)=X'(l)+\alpha X(l)=0, \alpha\in R. \quad (3.11)$$

If we introduce the notations

$$A_1(x)=\frac{-q(x)x^2+2p'(x)x+2p(x)}{2l+\alpha l^2},$$

$$A_2(x)=\frac{1+\alpha l}{2l+\alpha l^2}\left(\mathrm{q}(\mathrm{x})\mathrm{x}^2\text{-}2\mathrm{p}'(\mathrm{x})\mathrm{x}\text{-}2\mathrm{p}(\mathrm{x})+\frac{(\text{-}\mathrm{q}(\mathrm{x})\mathrm{x}+\mathrm{p}'(\mathrm{x}))(2l+\alpha l^2)}{1+\alpha l}\right),$$

$$A_3(x)=-\frac{\rho(\mathrm{x})\mathrm{x}^2}{2l+\alpha l^2},$$

$$A_4(x)=-\rho(\mathrm{x})\mathrm{x}+\frac{\rho(\mathrm{x})\mathrm{x}^2(1+\alpha l)}{2l+\alpha l^2},$$

then the Fourier coefficients $\gamma_{\mathrm{n}}(\mathrm{t})$ of the function $\gamma(t,x)$ can be represented in the form

$$\gamma_{\mathrm{n}}(\mathrm{t})=\int_0^l\left(\mathrm{A}_1(\mathrm{x})\cdot\eta_2(\mathrm{t})+\mathrm{A}_2(\mathrm{x})\cdot\eta_1(\mathrm{t})+\mathrm{A}_3(\mathrm{x})\cdot\eta_2'(\mathrm{t})+\mathrm{A}_4(\mathrm{x})\cdot\eta_1'(\mathrm{t})\right)\mathrm{X}_{\mathrm{n}}(\mathrm{x})d\mathrm{x}.$$

Introduce the following notations

$$\mathrm{S}_{00}(\mathrm{t},\mathrm{x})=\mathrm{z}(\mathrm{t},\mathrm{x}), \tag{3.12}$$

$$S_{01}(t,x)=\sum_{n=1}^{\infty}\mathrm{X}_{\mathrm{n}}'(\mathrm{x})\int_0^t\gamma_{\mathrm{n}}(u)\,\mathrm{e}^{-\lambda_{\mathrm{n}}(t-u)}d\mathrm{u}+2x\left(\frac{1}{2l+\alpha l^2}\eta_2(\mathrm{t})-\frac{1+\alpha l}{2l+\alpha l^2}\eta_1(\mathrm{t})\right), \tag{3.13}$$

$$S_{02}(t,x)=\sum_{n=1}^{\infty}\mathrm{X}_{\mathrm{n}}''(\mathrm{x})\int_0^t\gamma_{\mathrm{n}}(u)\,\mathrm{e}^{-\lambda_{\mathrm{n}}(t-u)}d\mathrm{u}+2\left(\frac{1}{2l+\alpha l^2}\eta_2(\mathrm{t})-\frac{1+\alpha l}{2l+\alpha l^2}\eta_1(\mathrm{t})\right), \tag{3.14}$$

$$S_{10}(t,x)=\sum_{n=1}^{\infty}\mathrm{X}_{\mathrm{n}}(\mathrm{x})\left(\gamma_{\mathrm{n}}(\mathrm{t})-\lambda_{\mathrm{n}}\int_0^t\gamma_{\mathrm{n}}(u)\,\mathrm{e}^{-\lambda_{\mathrm{n}}(t-u)}d\mathrm{u}\right)+\left(\frac{1}{2l+\alpha l^2}\eta_2(\mathrm{t})-\frac{1+\alpha l}{2l+\alpha l^2}\eta_1(\mathrm{t})\right)\mathrm{x}^2+\eta_1(\mathrm{t})\mathrm{x}. \tag{3.15}$$

With the help of these notations we can prove the following theorem.

Theorem 3.1 *For existing of twice continuously differentiable (two times with respect to variable x and one time with respect to variable t) solution of the problem (3.1)-(3.3) in probability of the form*

$$\begin{aligned}z(t,x)=&\left(\frac{1}{2l+\alpha l^2}\eta_2(\mathrm{t})-\frac{1+\alpha l}{2l+\alpha l^2}\eta_1(\mathrm{t})\right)\mathrm{x}^2+\eta_1(\mathrm{t})\mathrm{x}+\\&+\sum_{n=1}^{\infty}X_n(x)\Bigg(\int_0^t\eta_2(t')e^{-\lambda_n(t-t')}dt'\int_0^l A_1(x')X_n(x')dx'+\\&+\int_0^t\eta_1(t')e^{-\lambda_n(t-t')}dt'\int_0^l A_2(x')X_n(x')dx'+\int_0^t\eta_2'(t')e^{-\lambda_n(t-t')}dt'\int_0^l A_3(x')X_n(x')dx'+\\&\int_0^t\eta_1'(t')e^{-\lambda_n(t-t')}dt'\int_0^l A_4(x')X_n(x')dx'\Bigg),\end{aligned} \tag{3.16}$$

such that that the series (3.12)-(3.15) converge uniformly, it is sufficient that the series

$$\sum_{n=1}^{\infty}X_n(x)\int_0^l A_i(x')X_n(x')dx'; \tag{3.17}$$

converge in the norm of the space $C([0,l]\times[0,T])$, and the series

$$\sum_{n=1}^{\infty}\lambda_n X_n(x)\int_0^t \tau_i(t')e^{-\lambda_n(t-t')}dt'\int_0^l A_i(x')X_n(x')dx'; \qquad (3.18)$$

where $i=1,2,3,4;\tau_1(t)=\eta_2(t),\tau_2(t)=\eta_1(t),\tau_3(t)=\eta_2'(t),\tau_4(t)=\eta_1'(t);$ *converge in probability in the norm of the space* $C([0,l]\times[0,T])$ *and there exist* $\eta_1'(t),\eta_2'(t)$ *with probability one.*

Proof. In order to prove the statement of the theorem we need to reveal the uniform convergence in probability in the space $C([0,l]\times[0,T])]$) of the series (3.17) and the series

$$\sum_{n=1}^{\infty}X_n'(x)\int_0^t \tau_i(t')e^{-\lambda_n(t-t')}dt'\int_0^l A_i(x')X_n(x')dx, \qquad (3.19)$$

$$\sum_{n=1}^{\infty}X_n''(x)\int_0^t \tau_i(t')e^{-\lambda_n(t-t')}dt'\int_0^l A_i(x')X_n(x')dx; \qquad (3.20)$$

$$\sum_{n=1}^{\infty}X_n(x)\int_0^l A_i(x')X_n(x')dx\left(\tau_i(t)-\lambda_n\int_0^t \tau_i(t')e^{-\lambda_n(t-t')}dt'\right), \qquad (3.21)$$

where $i=1,2,3,4;\tau_1(t)=\eta_2(t),\tau_2(t)=\eta_1(t),\tau_3(t)=\eta_2'(t),\tau_4(t)=\eta_1'(t)$. Note, that summing up series (3.19)-(3.21) over all values of the variable $i = 1,2,3,4$, we obtain that the corresponding series are derivatives in t and x of the random function z(t, x) two times with respect to variable x and one time with respect to variable t. Uniform convergence in probability in the space $C([0,l]\times[0,T])$ of series (3.17), (3.19)-(3.21) ensures the existence of subsequences of these series, which converge uniformly with probability one. And this provides existence of solution (classical) of the problem (3.1)-(3.3).

Convergence (3.21) follows from the convergence of series (3.17), (3.18).

We know that (Polozhiy, 1964, p. 433)

$$X_k(x)=\lambda_k\int_0^l G(x,s)\rho(s)X_k(s)ds,$$

with

$$G(x,s)=\begin{cases}\frac{1}{\Delta}u(s)v(x), s\le x,\\ \frac{1}{\Delta}u(x)v(s), s>x,\end{cases}$$

where $u(x)$ and $v(x)$ are continuously differentiable functions, $\Delta>0$, $\Delta=p(x)(u'(x)v(x)-u(x)v'(x))$.

Let

$$X_k'(x)=\frac{\lambda_k}{\Delta}\int_0^l G^*(x,s)\rho(s)X_k(s)ds,$$

where

$$\mathrm{G}^*(\mathrm{x},\mathrm{s})=\begin{cases}\frac{1}{\Delta}\mathrm{u}(\mathrm{s})\mathrm{v}'(\mathrm{x}), s\le x,\\ \frac{1}{\Delta}\mathrm{u}'(x)\mathrm{v}(s), s>x,\end{cases}$$

Consider (3.19) for $i = 1$. In this case

$$\sum_{k=n}^{m} X_k'(x)\int_0^t \eta_2(t')e^{-\lambda_k(t-t')}dt'\int_0^l A_1(x')X_k(x')dx' =$$

$$=\frac{1}{\Delta}\int_0^l\left(G^*(x,s)\rho(s)\sum_{k=n}^{m}\lambda_k X_k(s)\int_0^t \eta_2(t')e^{-\lambda_k(t-t')}dt'\int_0^l A_1(x')X_k(x')dx'\right)ds,$$

and $\forall\delta > 0\ \exists N$ such that for all $n > N$:

$$P\left\{\sup_{\substack{x\in[0,l]\\ t\in[0,T]}}\left|\sum_{k=n}^{m} X_k'(x)\int_0^t \eta_2(t')e^{-\lambda_k(t-t')}dt'\int_0^l A_1(x')X_k(x')dx'\right|>\delta\right\}\le$$

$$=P\left\{\sup_{\substack{x\in[0,l]\\ t\in[0,T]}}\left|\frac{1}{\Delta}\int_0^l\left(G^*(x,s)\rho(s)\sum_{k=n}^{m}\lambda_k X_k(s)\int_0^t \eta_2(t')e^{-\lambda_k(t-t')}dt'\int_0^l A_1(x')X_k(x')dx'\right)ds,\right|>\delta\right\}\le$$

$$\le P\left\{C^*\cdot\sup_{\substack{x\in[0,l]\\ t\in[0,T]}}\left|\sum_{k=n}^{m}\lambda_k X_k(s)\int_0^t \eta_2(t')e^{-\lambda_k(t-t')}dt'\int_0^l A_1(x')X_k(x')dx'\right|>\delta\right\}\to 0,$$

as $m,n \to \infty$, where

$$C^*=\frac{1}{\Delta}\int_0^l G^*(l,s)\rho(s)ds.$$

Therefore the series converges in probability in the space $C([0,l]\times[0,T])$. In the same way we get convergence in probability in the space $C([0,l]\times[0,T])$ of the series for $i = 2,3,4$.
Consider

$$\mathrm{X}_k''(\mathrm{x})=\frac{\lambda_k}{\Delta}\left[\int_0^l \mathrm{G}^*(\mathrm{x},\mathrm{s})\rho(\mathrm{s})\mathrm{X}_k(\mathrm{s})\mathrm{ds}+\left(v'(x)u(x)-u'(x)v(x)\right)\rho(x)X_k(x)\right],$$

where

$$\mathrm{G}^{**}(\mathrm{x},\mathrm{s})=\begin{cases}\frac{1}{\Delta}\mathrm{u}(\mathrm{s})\mathrm{v}''(\mathrm{x}), s\le x,\\ \frac{1}{\Delta}\mathrm{u}''(x)\mathrm{v}(s), s>x,\end{cases}$$

Then the series (3.20) for $i = 1$ can be represented in the form

$$\sum_{k=n}^{m} X_k''(x)\int_0^t \eta_2(t')e^{-\lambda_k(t-t')}dt'\int_0^l A_1(x')X_k(x')dx' =$$

$$=\frac{1}{\Delta}\int_0^l\left(G^{**}(x,s)\rho(s)\sum_{k=n}^{m}\lambda_k X_k(s)\int_0^t\eta_2(t')e^{-\lambda_k(t-t')}dt'\int_0^l A_1(x')X_k(x')dx'\right)ds+$$

$$+\frac{1}{\Delta}\big(v'(x)u(x)-u'(x)v(x)\big)\rho(x)\sum_{k=n}^{m}\lambda_k X_k(x)\int_0^t\eta_2(t')e^{-\lambda_k(t-t')}dt'\int_0^l A_1(x')X_k(x')dx'.$$

This series converges in probability in the space $C([0,l]\times[0,T])$. In the same way we get the convergence in probability in the space $C([0,l]\times[0,T])$ of the series for $i=2,3,4$.
Consider

$$S_i^{**}(t,x)=\sum_{k=1}^{\infty}\lambda_k X_k(x)\int_0^t\tau_i(t')e^{-\lambda_k(t-t')}dt'\int_0^l A_i(x')X_k(x')dx';$$

$$S_{in}^{**}(t,x)=\sum_{k=1}^{n}\lambda_k X_k(x)\int_0^t\tau_i(t')e^{-\lambda_k(t-t')}dt'\int_0^l A_i(x')X_k(x')dx';$$

where $\mathrm{i}=1,2,3,4;\ \tau_1(\mathrm{t})=\eta_2(\mathrm{t}),\tau_2(\mathrm{t})=\eta_1(\mathrm{t}),\tau_3(\mathrm{t})=\eta_2'(\mathrm{t}),\tau_4(\mathrm{t})=\eta_1'(\mathrm{t})$.

Lemma 3.1 *Let the series*

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\lambda_k\lambda_n X_k(x)X_n(x)\int_0^t\int_0^t e^{-\lambda_k(t-t')}e^{-\lambda_n(t-t'')}E\tau_i(t')\tau_i(t'')\int_0^l\int_0^l A_i(x')A_j(x'')X_k(x')X_n(x'')dx'dx''dt'dt'', \quad (3.22)$$

convergence in probability for *any* $x\in[0,l], t\in[0,T]$, *where* $i=1,2,3,4;\ j=1,2,3,4;$ *(*$i=j$ *or* $i=j-2$ *or* $i=j+2$ *)* $\tau_1(\mathrm{t})=\eta_2(\mathrm{t}),\tau_2(\mathrm{t})=\eta_1(\mathrm{t}),\tau_3(\mathrm{t})=\eta_2'(\mathrm{t}),\tau_4(\mathrm{t})=\eta_1'(\mathrm{t})$. If *the following conditions are satisfied*

$$\sup_{n\ge 1}\sup_{\substack{|x-y|\le h\\|t-s|\le h}}\left(E\left|S_{in}^{**}(t,x)-S_{in}(s,y)\right|^2\right)^{\frac{1}{2}}\le\sigma_{i^{**}}(h), \quad (3.23)$$

where $\sigma_{i^{**}}=\{\sigma_{i^{**}}(\mathrm{h}),\mathrm{h}>0\}$ *are continuous monotonically increasing functions, and* $\sigma_{i^{**}}(\mathrm{h})\to 0$ *as* $h\to 0$ *and for some* $\varepsilon>0$

$$\int_0^{\varepsilon}U^{(-1)}\left(\left(\frac{T}{2\sigma_{i^{**}}^{(-1)}(u)}+1\right)\left(\frac{l}{2\sigma_{i^{**}}^{(-1)}(u)}+1\right)\right)du<\infty,$$

where $\sigma_{i^{**}}^{(-1)}(\mathrm{u})$ *is the inverse functions to* $\sigma_{i^{**}}(\mathrm{u})$. *Then (3.18) converge in the norm of the space* $C([0,l]\times[0,T])$.

Proof. This lemma is a corollary of Theorem 2.2. Since the functions $\mathrm{X_k}(\mathrm{x})$ are continuous, then the functions $S_{\mathrm{in}}^{**}(\mathrm{t,x})$ are sample continuous with probability one, and these functions $S_{\mathrm{in}}^{**}(\mathrm{t,x})$ are separable. Condition (3.22) provides convergence $S_{\mathrm{in}}^{**}(\mathrm{t,x})$ in the mean square, and hence in probability. Conditions 3), 4) of Theorem 2.2 follow from (3.23).

Lemma 3.2 *Consider the problem (3.1)-(3.3). Let* $\eta_2(\mathrm{t})$, $\eta_1(\mathrm{t})$ *be independent strictly Orlicz stochastic processes belonging to the Orlicz space* $\mathrm{L_U}(\Omega)$, *and let condition g holds true for* $U(x)$. *Let there exist a continuous increasing function* $\varphi(\lambda)(\lambda>0)$ *such that* $\varphi(\lambda)>0$ *and* $\frac{\lambda}{\varphi(\lambda)}$ *increases for* $\lambda > v_0 (v_0 = const, v_0 \ge 0)$. *Let*

$$C_{k,m} = \sup_{i,j\in\overline{1,4}} \sup_{t,t_1\in[0,T]} \left| E\tau_i(t)\tau_j(t)\int_0^l\int_0^l A_i(x')A_j(x'')X_k(x')X_m(x'')dx'dx'' \right|,$$

where $i = 1,2,3,4$; $i = 1,2,3,4$; ($i = j$ *or* $i = j-2$ *or* $i = j+2$) $\tau_1(\mathrm{t}) = \eta_2(\mathrm{t}), \tau_2(\mathrm{t}) = \eta_1(\mathrm{t}), \tau_3(\mathrm{t}) = \eta_2'(\mathrm{t}), \tau_4(\mathrm{t}) = \eta_1'(\mathrm{t})$. *Let us also assume that the series*

$$F_1 = \left(\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} C_{k,m}\varphi(\lambda_k+v_0)\varphi(\lambda_m+v_0) \right)^{\frac{1}{2}} \left(\max(2C_x, L) + 3C_x \right)$$

converges and satisfies condition (3.22). Then for $n \ge 1$

$$\sup_{\substack{|x-x_1|\le h \\ |t-t_1|\le h \\ x,x_1\in[0,l] \\ t,t_1\in[0,T]}} \left(E\left| S_{in}^{**}(t,x) - S_{in}(t_1,x_1) \right|^2 \right)^{\frac{1}{2}} \le F_1 \left(\varphi\left(\frac{1}{h} + v_0 \right) \right)^{-1},$$

where

$$S_{in}^{**}(t,x) = \sum_{k=1}^{n} \lambda_k X_k(x) \int_0^t \tau_i(t') e^{-\lambda_k(t-t')}dt' \int_0^l A_i(x')X_k(x')dx';$$

$$L = \left(C_u' C_v + 2C_u C_v + C_u C_v' \right) C_x,$$

$$C_u = \sup_{0\le x\le l} |u(x)|, C_u' = \sup_{0\le x\le l} |u'(x)|,$$

$$C_v = \sup_{0\le y\le l} |v(y)|, C_v' = \sup_{0\le y\le l} |v'(y)|,$$

($u(x), v(y)$ *— defined in Theorem 3.1),* C_x *a constant, such that* $|X_k(x)| \le C_x$.

Proof. Let $|\mathrm{x}-\mathrm{x}_1| \le \mathrm{h}$, $|\mathrm{t}-\mathrm{t}_1| \le \mathrm{h}$, $\mathrm{x},\mathrm{x}_1 \in [0,l]$, $\mathrm{t},\mathrm{t}_1 \in [0,\mathrm{T}]$. Consider

$$\left(E\left| S_{in}^{**}(t,x) - S_{in}(t_1,x_1) \right|^2 \right)^{\frac{1}{2}} =$$

$$= \left(E\left| \sum_{k=1}^{n} \lambda_k X_k(x) \int_0^t \tau_i(t') e^{-\lambda_k(t-t')}dt' \int_0^l A_i(x')X_k(x')dx' - \right.\right.$$

$$\left.\left. - \sum_{k=1}^{n} \lambda_k X_k(x_1) \int_0^{t_1} \tau_i(t') e^{-\lambda_k(t_1-t')}dt' \int_0^l A_i(x')X_k(x')dx' \right|^2 \right)^{\frac{1}{2}} \le$$

$$\le \left(E\left| \sum_{k=1}^{n} \lambda_k \left(X_k(x) - X_k(x_1) \right) \int_0^t \tau_i(t') e^{-\lambda_k(t-t')} dt' \int_0^l A_i(x') X_k(x') dx' \right|^2 \right)^{\frac{1}{2}} +$$

$$+ \left(E\left| \sum_{k=1}^{n} \lambda_k X_k(x_1) \left(\int_0^t \tau_i(t') e^{-\lambda_k(t-t')} dt' \int_0^l A_i(x') X_k(x') dx' - \right.\right.\right.$$

$$\left.\left.\left. - \int_0^{t_1} \tau_i(t') e^{-\lambda_k(t_1-t')} dt' \int_0^l A_i(x') X_k(x') dx' \right|^2 \right)^{\frac{1}{2}} = E_1 + E_2.$$

Denote

$$R_{km}(t,t_1) = \int_0^t \int_0^{t_1} e^{-\lambda_k(t-t')} e^{-\lambda_m(t_1-t'')} E\tau_i(t')\tau_j(t'') \int_0^l \int_0^l A_i(x') A_j(x'') X_k(x') X_m(x'') dx' dx'' dt' dt''.$$

Then

$$E_1^{\,2} \le \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \lambda_k \lambda_m \left| X_k(x) - X_k(x_1) \right| \left| X_m(x) - X_m(x_1) \right| R_{km}(t,t).$$

It follows from conditions of Lemma 2.1 that

$$\left| X_l(x) - X_l(x_1) \right| \le \max(2C_x, L) \frac{\varphi(\lambda_l + v_0)}{\varphi\left(\frac{1}{h} + v_0 \right)},$$

where $h = x - x_1$. The following conditions are satisfied

$$\sup_{u \in [0,l]} \left| X_k(u) \right| \le C_x,$$

$$\left| X_k(x_1) - X_k(x_2) \right| \le L\lambda_k \left| x_1 - x_2 \right|,$$

where $x_1, x_2 \in [0,l]$ (see Kozachenko and Veresh, 2010). We have

$$E_1^{\,2} \le \left(\frac{\max(2C_x, L)}{\varphi\left(\frac{1}{h} + v_0 \right)} \right)^2 \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} C_{k,m} \varphi(\lambda_k + v_0) \varphi(\lambda_m + v_0).$$

Let $t_1 < t$. Then

$$\left| Q_{km}(t,t_1) \right| = E\left| \left(\int_0^t \tau_i(t') e^{-\lambda_k(t-t')} dt' \int_0^l A_i(x') X_k(x') dx' - \int_0^{t_1} \tau_i(t') e^{-\lambda_k(t_1-t')} dt' \int_0^l A_i(x') X_k(x') dx' \right) \cdot \right.$$

$$\left. \cdot \left(\int_0^t \tau_i(t') e^{-\lambda_m(t-t')} dt' \int_0^l A_i(x') X_m(x') dx' - \int_0^{t_1} \tau_i(t') e^{-\lambda_m(t_1-t')} dt' \int_0^l A_i(x') X_m(x') dx' \right) \right| =$$

$$= E\left|\left(\int_0^{t_1}\tau_i(t')\left(e^{-\lambda_k(t-t')}-e^{-\lambda_k(t_1-t')}\right)dt'\int_0^l A_i(x')X_k(x')dx' - \int_{t_1}^{t}\tau_i(t')e^{-\lambda_k(t-t')}dt'\int_0^l A_i(x')X_k(x')dx'\right)\cdot\right.$$
$$\left.\cdot\left(\int_0^{t_1}\tau_i(t')\left(e^{-\lambda_m(t-t')}-e^{-\lambda_m(t_1-t')}\right)dt'\int_0^l A_i(x')X_m(x')dx' - \int_{t_1}^{t}\tau_i(t')e^{-\lambda_m(t-t')}dt'\int_0^l A_i(x')X_m(x')dx'\right)\right| =$$
$$= \left|\int_0^{t_1}\int_0^{t_1}E\tau_i(t')\tau_j(t'')\int_0^l\int_0^l A_i(x')A_j(x'')X_k(x')X_m(x'')dx'dx''\left(e^{-\lambda_k(t-t')}-e^{-\lambda_k(t_1-t')}\right)\cdot\right.$$
$$\left(e^{-\lambda_m(t-t'')}-e^{-\lambda_m(t_1-t'')}\right)dt'dt''+$$
$$+\int_0^{t_1}\int_{t_1}^{t}E\tau_i(t')\tau_j(t'')\int_0^l\int_0^l A_i(x')A_j(x'')X_k(x')X_m(x'')dx'dx''\left(e^{-\lambda_k(t-t')}-e^{-\lambda_k(t_1-t')}\right)e^{-\lambda_m(t-t'')}dt'dt''+$$
$$+\int_{t_1}^{t}\int_0^{t_1}E\tau_i(t')\tau_j(t'')\int_0^l\int_0^l A_i(x')A_j(x'')X_k(x')X_m(x'')dx'dx''e^{-\lambda_k(t-t')}\left(e^{-\lambda_m(t-t'')}-e^{-\lambda_m(t_1-t'')}\right)dt'dt''+$$
$$\left.+\int_{t_1}^{t}\int_{t_1}^{t}E\tau_i(t')\tau_j(t'')\int_0^l\int_0^l A_i(x')A_j(x'')X_k(x')X_m(x'')dx'dx''e^{-\lambda_k(t-t')}e^{-\lambda_m(t-t'')}dt'dt''\right| \le$$
$$\le C_{k,m}\left(\int_0^{t_1}\int_0^{t_1}\left(e^{-\lambda_k(t_1-t')}-e^{-\lambda_k(t-t')}\right)\cdot\left(e^{-\lambda_m(t_1-t'')}-e^{-\lambda_m(t-t'')}\right)dt'dt''+\right.$$
$$+\int_0^{t_1}\int_{t_1}^{t}\left(e^{-\lambda_k(t_1-t')}-e^{-\lambda_k(t-t')}\right)\cdot e^{-\lambda_m(t-t'')}dt'dt''+$$
$$\left.+\int_{t_1}^{t}\int_0^{t_1}e^{-\lambda_k(t-t')}\cdot\left(e^{-\lambda_m(t_1-t'')}-e^{-\lambda_m(t-t'')}\right)dt'dt''+\int_{t_1}^{t}\int_{t_1}^{t}e^{-\lambda_k(t-t')}\cdot e^{-\lambda_m(t-t'')}dt'dt''\right) \le$$
$$\le \frac{9C_{k,m}\varphi(\lambda_k+v_0)\varphi(\lambda_m+v_0)}{\lambda_k\lambda_m\varphi^2\left(\frac{1}{t-t_1}+v_0\right)}.$$

Therefore

$$E_2^{\,2} \le \left(\frac{3C_x}{\varphi\left(\frac{1}{h}+v_0\right)}\right)^2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}C_{k,m}\varphi(\lambda_k+v_0)\varphi(\lambda_m+v_0). \qquad \square$$

In the theorem 3.2 we describe conditions under which the boundary value problem for parabolic equations (3.1)-(3.3) has a solution which is represented in the form *(3.16).*

Theorem 3.2 *Consider the problem (3.1)-(3.3). Let $\eta_2(t)$, $\eta_1(t)$ be independent strictly Orlicz stochastic processes belonging to the Orlicz space $L_U(\Omega)$, and let condition g holds true for $U(x)$. Let there exist a continuous increasing function $\varphi(\lambda)(\lambda>0)$ such that $\varphi(\lambda)>0$ and $\frac{\lambda}{\varphi(\lambda)}$ increases for $\lambda>v_0(v_0=const, v_0\geq 0)$. Let the following series converge*

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}C_{k,m}\varphi(\lambda_k+v_0)\varphi(\lambda_m+v_0),$$

where

$$C_{k,m}=\sup_{i,j\in\overline{1,4}}\sup_{t,t_1\in[0,T]}\left|E\tau_i(t)\tau_j(t)\int_0^l\int_0^l A_i(x')A_j(x'')X_k(x')X_m(x'')dx'dx''\right|,$$

$i=1,2,3,4$; $j=1,2,3,4$; ($i=j$ or $i=j-2$ or $i=j+2$) $\tau_1(t)=\eta_2(t)$, $\tau_2(t)=\eta_1(t), \tau_3(t)=\eta_2'(t), \tau_4(t)=\eta_1'(t)$. Moreover, let for $\forall\varepsilon>0$ the following integral converges

$$\int_0^{\varepsilon}U^{(-1)}\left(\left(\frac{l}{2}\left(\varphi^{(-1)}\left(\frac{F_1}{v}\right)-v_0\right)+1\right)\left(\frac{T}{2}\left(\varphi^{(-1)}\left(\frac{F_1}{v}\right)-v_0\right)+1\right)\right)dv<\infty, \qquad (3.24)$$

where

$$F_1=\left(\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}C_{k,m}\varphi(\lambda_k+v_0)\varphi(\lambda_m+v_0)\right)^{\frac{1}{2}}\left(\max(2C_x,L)+3C_x\right).$$

$$L=\left(C_u'C_v+2C_uC_v+C_uC_v'\right)C_x,$$

$$C_u=\sup_{0\leq x\leq l}|u(x)|, C_u'=\sup_{0\leq x\leq l}|u'(x)|,$$

$$C_v=\sup_{0\leq y\leq l}|v(y)|, C_v'=\sup_{0\leq y\leq l}|v'(y)|,$$

($u(x), v(y)$ are defined in Theorem 3.1), C_x is a constant, such that $|X_k(x)|\leq C_x$.

Then series (3.18) converges uniformly in probability in $C([0,t]\times[0,l])$ and there exists a solution of the problem (3.1)-(3.3), which is represented in the form (3.16), such that series (3.19), (3.20), (3.21) converge uniformly in probability.

Proof. This theorem follows from lemma 3.1 and lemma 3.2 if we take

$$\sigma_{i^{**}}(h)=F_1\left(\varphi\left(\frac{1}{h}+v_0\right)\right)^{-1}. \qquad \square$$

Lemma 3.3 *Consider the problem (3.1)-(3.3). Let $X_k(x)$, $x\in[0,l]$ be eigenfunctions of the Sturm-Liouville problem (3.10), (3.11). Let the following series converge*

$$W=\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{C_{k,m}}{\lambda_k\lambda_m}\varphi(\lambda_k+v_0)\varphi(\lambda_m+v_0),$$

where φ is defined in Lemma 3.2,

$$C_{k,m} = \sup_{i,j\in\overline{1,4}} \sup_{t,t_1\in[0,T]} \left| E\tau_i(t)\tau_j(t)\int_0^l\int_0^l A_i(x')A_j(x'')X_k(x')X_m(x'')dx'dx'' \right|,$$

$i = 1,2,3,4$; $j = 1,2,3,4$; *(* $i = j$ *or* $i = j-2$ *or* $i = j+2$ *)* $\tau_1(t)=\eta_2(t)$, $\tau_2(t)=\eta_1(t)$, $\tau_3(t)=\eta_2'(t), \tau_4(t)=\eta_1'(t)$, $L=(C_u'C_v+2C_uC_v+C_uC_v')C_x$,

$$C_u = \sup_{0\le x\le l}|u(x)|, C_u' = \sup_{0\le x\le l}|u'(x)|,$$

$$C_v = \sup_{0\le y\le l}|v(y)|, C_v' = \sup_{0\le y\le l}|v'(y)|,$$

($u(x), v(y)$ are defined in Theorem 3.1), C_x *is a constant, such that* $|X_k(x)|\le C_x$.

Then for $t_1,t_2\in[0,T]$, $x_1,x_2\in[0,l]$ *the following condition is satisfied*

$$\sup_{\max(|t_1-t_2|,|x_1-x_2|)\le h}\left(E\left(z_N(t_1,x_1)-z_N(t_2,x_2)\right)^2\right)^{1/2} \le \frac{4\max(C_x,L)+12C_x}{\varphi\left(\frac{1}{h}+v_0\right)}\sqrt{W},$$

where

$$z_N(t,x)=\sum_{n=N}^{\infty}X_n(x)\left(\int_0^t \eta_2(t')e^{-\lambda_n(t-t')}dt'\int_0^l A_1(x')X_n(x')dx' + \right.$$

$$+\int_0^t \eta_1(t')e^{-\lambda_n(t-t')}dt'\int_0^l A_2(x')X_n(x')dx' + \int_0^t \eta_2'(t')e^{-\lambda_n(t-t')}dt'\int_0^l A_3(x')X_n(x')dx' +$$

$$\left. +\int_0^t \eta_1'(t')e^{-\lambda_n(t-t')}dt'\int_0^l A_4(x')X_n(x')dx'\right).$$

Proof. Since (see Kozachenko and Veresh, 2010)

$$|X_k(x_1)-X_k(x_2)|\le L\lambda_k|x_1-x_2|,$$

then for $|t_1-t_2|\le h$, $|x_1-x_2|\le h$ (see Lemma 3.2)

$$\left(E\left(z_N(t_1,x_1)-z_N(t_2,x_2)\right)^2\right)^{1/2} =$$

$$\left(E\left(\sum_{i=1}^{4}\sum_{n=N}^{\infty}X_n(x_1)\int_0^{t_1}\tau_i(t')e^{-\lambda_n(t_1-t')}dt'\int_0^l A_i(x')X_n(x')dx' - \right.\right.$$

$$\left.\left.-\sum_{i=1}^{4}\sum_{n=N}^{\infty}X_n(x_2)\int_0^{t_2}\tau_i(t')e^{-\lambda_n(t_2-t')}dt'\int_0^l A_i(x')X_n(x')dx'\right)^2\right)^{1/2} \le$$

$$\le \frac{4\max(C_x,L)}{\varphi\left(\frac{1}{h}+v_0\right)}\sqrt{\sum_{k=N}^{\infty}\sum_{l=N}^{\infty}\frac{C_{k,l}}{\lambda_k\lambda_l}\varphi(\lambda_k+v_0)\varphi(\lambda_l+v_0)} +$$

$$\frac{12C_x}{\varphi\left(\frac{1}{h}+v_0\right)}\sqrt{\sum_{k=N}^{\infty}\sum_{l=N}^{\infty}\frac{C_{k,l}}{\lambda_k\lambda_l}\varphi(\lambda_k+v_0)\varphi(\lambda_l+v_0)} =$$

$$= \frac{4\max(C_x, L)+12C_x}{\varphi\left(\frac{1}{h}+v_0\right)}\sqrt{W}.$$

□

In theorem 3.3 we describe estimates for distribution of supremum of solution of the boundary value problem for parabolic equations (3.1)-(3.3).

Theorem 3.3 *Consider the problem (3.1)-(3.3). Let conditions of Lemma 3.3 be satisfied. Then for any* $\varepsilon > 0$

$$P\left\{\sup_{\substack{0\le x\le l\\ 0\le t\le\hat{t}<T}}|z_N(t,x)|>\varepsilon\right\}\le\left(U\left(\frac{\varepsilon}{B_N(t_0,x_0)}\right)\right)^{-1}, \tag{3.25}$$

where $x_0\in[0,l], t_0\in[0,T)$,

$$\left\|\sup_{\substack{0\le x\le l\\ 0\le t\le\hat{t}<T}}|z_N(t,x)|\right\|_{L_U}\le B_N(t_0,x_0)=\|z_N(t_0,x_0)\|+$$

$$+\frac{C_\Delta}{\theta(1-\theta)}\int_0^{\omega_{0N}\theta}U^{(-1)}\left(\left(\frac{l}{2}\left(\varphi^{(-1)}\left(\frac{R_N}{v}\right)-v_0\right)+1\right)\left(\frac{T}{2}\left(\varphi^{(-1)}\left(\frac{R_N}{v}\right)-v_0\right)+1\right)\right)dv<\infty,$$

$$0<\theta<1, R_N=2\sqrt{W}(4\max(C_x,L)+12C_x),$$

$$\omega_{0N}=\frac{R_N}{\varphi\left(\frac{1}{\max(T,l)+v_0}\right)}.$$

Proof. The proof of the statement follows from Theorem 2.3 and Corollary 2.1. In this theorem

$$\sigma(h)=\frac{4\max(C_x,L)+12C_x}{\varphi\left(\frac{1}{h}+v_0\right)}\sqrt{W}$$

and condition (3.24) is satisfied.

□

4. Concluding Remarks

Conditions justifying application of the Fourier method for parabolic equations with boundary conditions that are stochastic processes belonging to the Orlicz spaces of random variables are obtained in the paper. Some bounds for the distribution of the supremum of solutions of such equations are found.

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