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Research Article

Optimal Maneuver of Mechanical System with Internal Degrees of Freedom with State-Variable Constraint

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Abstract

An effective procedure is presented for the determination of the optimal control input for maneuvers of a mechanical system with internal degrees of freedom such as a slewing of a spacecraft with flexible appendages or a displacement of a reservoir with a liquid for the case of constraint on velocity of the maneuver. The dynamic equations of motion are formulated, allowing taking into account the flexible elements using the quasistatical approach. The problem of optimal reorientation for rest-to-rest maneuvers is formulated using the objective function, which results in the minimal acceleration of the relative motion of the attached flexible elements during the maneuver. The new features and advantages of the proposed approach are the use of a not widely known objective function for the optimal control problem, which has a clear physical interpretation, and analytical solving the constrained optimization problem by the method based on parameterization of the functional for the multi-point boundary value problem. The solution is illustrated graphically. This analytical solution is applicable also for vibrations rejection at shaping the law of deployment of flexible constructions on spacecraft. It is useful for input shaping motion laws of objects of the ground-based transport in the modes of braking and acceleration for minimization of relative accelerations of passengers and goods.

Keywords: Mechanical system; Internal degrees of freedom; Rest-to-rest maneuver; Optimal control; Constraint

1. Introduction

Optimization of the rest-to-rest maneuver of a spacecraft with attached elastic elements attracts attention of researchers for a long time. Such maneuvers are the main modes of operation of spacecraft. This interest does not relax now since there are new types of the constructions, new actuators and sensors, new types of the control systems.

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In practice, attitude control of such systems uses a combination of a feed forward and feedback control. Measurement of the deformation of flexible appendages is possible only if distributed sensors are used. The performance of the feedback is limited, if a spacecraft does not have such sensors. Thus, the feed forward control, which takes proper account of object dynamic properties, is of great importance for this problem. Optimization of a rest-to-rest maneuver of flexible spacecraft can entail various objectives. For SC on long missions with a weak power-to-mass ratio, an important objective is to minimize the energy cost (VanderVelde and He, 1983; Wie et al., 1993; Singh, 1995; Meyer and Silverberg, 1996). For spacecraft dealing with astronomical observations and performing a great number of reorientation maneuvers, it may be important to minimize time. Time-optimal rest-to-rest slewing of flexible systems has been the subject of much research (Bryson and Ho, 1969; Singh G. et al, 1989; Bainum and Li, 1990; Ben-Asher et al, 1992; Liu and Wie, 1992; Singh T. and Vadali, 1993; Pao, 1996; Di Meglio and Finzi,1997; Singhose et al, 1996; Singhose et al, 1997; Banerjee et al, 2001). Typically, the solution of the problem results in a bang-bang control, which can be very sensitive to system modeling errors. Often, these objectives are combined into a single cost function with weighted items (Gorinevsky and Vukovich, 1997).

For some cases, minimization of the energy cost or the transfer time is not as important as the minimization of the perturbation of the main body's attitude motion by flexible vibrations. The slewing rest-to-rest maneuver of such spacecraft needs to be controlled so that practically no flexible vibrations of their appendages are excited during the maneuver.

The first attempts to formulate the objective function that takes into account design's elastic properties, were made in the seventies (Zakrzhevskii, 1972; Zakrzhevskii, 1977; Farrenkopf, 1979). This cost functions contain various combinations of generalized co-ordinates of relative elastic displacements and their derivatives. Other older investigations in this area include a number of approaches developed for linear flexible systems, which shape the feed forward input such that it does not contain spectral components at system Eigen-frequencies (Bayo et al, 1989). Modifications of such methods have been applied to nonlinear flexible systems (Singh and Vadali, 1993), but they may yield a significant level of residual vibrations (Gorinevsky and Vukovich, 1997).

All these papers treat elastic systems discretized only by one mode. Zakrzhevskii (2008) used a quadratic objective function, the physical meaning of which was to minimize the relative acceleration of the flexible elements during the controlled motion. The method allows considering infinite number of elastic modes not being beyond the finite-dimensional mathematical model. It appears possible, if to consider the first some modes in the traditional posing and all the higher ones to consider in quasistatic posing. Special case (Zakrzhevskii, 2008), when all flexible modes may be considered in quasistatic posing, represents separate interest for practice. It is possible either when the lowest frequencies of the spacecraft are enough high, or for flexible spacecraft with the high performance of the feedback control.

The purpose of this publication is to consider the problem of optimal feed forward control for a slewing maneuver of quasistatically deformable system that minimizes relative accelerations of elastic appendages when angular velocity of the slew is constrained.

2. Problem Formulation

Modes of reorientation of spacecraft can consist in orientation change both all three axes of the frame of reference connected with the main body, and one of the main axes. The last takes place when the registering device whose axis is directed along a reoriented one, accepts or transfers unpolarized signal. If to consider reorientation of one of principal axes of a spacecraft, the optimal control problem becomes linear one with a quadratic cost function, since problem statement does not consider the equations of motion of a main body. It is supposed *a priori* that the slewing of this body occurs in the mode of rotation around an axis of finite rotation (Eulerian axis). The equations of motion of the attached elastic elements become linear at this.

For definiteness, the axis Ox_1 can be chosen as reoriented one (**Fig 1**). Let it be necessary to redirect the spacecraft's axis Ox_1 from a direction Ox_{10} to a direction Ox_{1f}. The redirections of the axes Ox_2, Ox_3 can be arbitrary at this. The problem of determination of the axis of finite rotation does not have a unique solution in this case. It is reasonable to choose this axis such that the angle of finite rotation is minimal. This occurs if the axis of finite rotation is orthogonal to the initial and terminal positions of the reoriented axis (i.e., if it lies in one of the principal inertia planes of the unstrained spacecraft). In **Fig 1**, this is line $n-n'$.

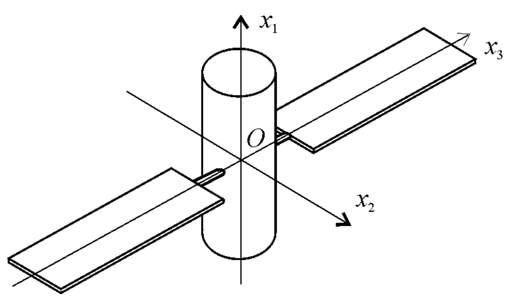

Fig. 1. Mechanical model of a spacecraft with flexible appendages

In this case, the axis of finite rotation yielding the minimal slewing angle lies in the plane x_2Ox_3. The optimal control problem becomes (Zakrzhevskii, 2008):

$$\dot{x}_1 = x_2;\ \ \dot{x}_2 = x_3;\ \ \dot{x}_3 = x_4;\ \ \dot{x}_4 = v; \tag{1}$$

$$I^* = \frac{1}{2}\int_0^T v^2 dt; \tag{2}$$

$$x_1(0) = -\Phi;\ \ x_i(0) = 0;\ \ x_j(T) = 0\ \ (i = 2,3,4;\ j = 1,\dots,4). \tag{3}$$

Here x_1 is the slewing angle of the spacecraft with respect to the axis of finite rotation, v is the control function.

For the case of no constraints the solution of the problem looks like:

$$x_i = \sum_{k=i-1}^{7} C_k t^{(k-i+1)}(k-i+1)!^{-1};\quad v = \sum_{k=4}^{7} C_k t^{(k-4)}(k-4)!^{-1}. \tag{4}$$

At boundary conditions (3), the integration constants are respectively

$$C_0 = C_1 = C_2 = C_3 = 0;$$

$$C_4 = 4!35\Phi T^{-4};\, C_5 = -5!84\Phi T^{-5};\; C_6 = 6!70\Phi T^{-6};\; C_7 = -7!20\Phi T^{-7}.. \tag{5}$$

The solution for $x_2(t)$ (slewing angular velocity) can be written as

$$\mathrm{x}_2(t) = -140\Phi t^6/T^7 + 420\Phi t^5/T^6 - 420t^4/T^5 + 140\Phi t^3/T^4. \tag{6}$$

If to study this expression, it is easy to see that it has the extremum at $t = T/2$. From the expression $\mathrm{x}_2(\mathrm{T}/2) = l$ it follows that the maximum slewing angle, which can be reached by the control at such a value of $x_2(T/2)$, is Φ_1=16/35 T l. Note that value $\mathrm{x}_4(\mathrm{T}/2)$ is equal to $-24l/T^2$ at that.

Let's consider now the problem (1) – (3) for the case of constraint on angular velocity of the slew:

$$S = |x_2| - l \le 0. \tag{7}$$

For c1arity and without any loss of generality one can assume that $\Phi > 0$. For the problem (1) – (3) in the idealized case of no constraints, solution $x_2(t)$ for state-variable inequality constrained problem (1) – (3) is shown in **Fig 2** (curve 1). When the value $x_1(0)$ decreases to some value $-\Phi_1$ curve $x_2(t)$ touches the constraint in the point $t = 0,5T$ (curve 2). At that $x_2(t) \ge 0\ \forall t \in [0, T]$.

Hence, one can replace the constraint (7) by more simple condition

$$S = x_2 - l \le 0. \tag{8}$$

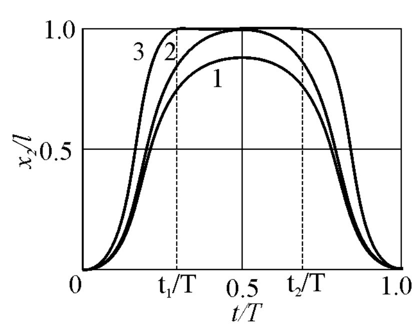

Fig. 2. Forms of solutions for $x_2(t)$

State-variable inequality constrained problems were studied by many authors. The main objective was to obtain the necessary conditions of optimality (Gamkrelidze, 1960; Bryson and Ho, 1969; Jacobson et al, 1969). Necessary conditions of optimality come usually to so-called tangential conditions in junction points of boundary arcs. Remaining conditions in these points are obtained by equaling to zero the first variations of the extended minimized functional written in the appropriate way.

For the phase variables unconstrained by tangential conditions, it gives linear ratio for the proper costate variables computed at the left and to the right of the junction points. Conditions are formed in each junction point for determination of the unknown values t_i. At the regular Hamiltonian, these conditions come to continuity of the control variable in these points. In any case, even in the linear problems with a quadratic cost function, it leads to the set of the nonlinear algebraic equations.

These difficulties can be overcome simply enough if to come to the problem in a little different way. There is the approach where the problem with a free terminal time is replaced by the sequence of the problems with the fixed terminal time (Bryson and Ho, 1969). In other words, it is possible to regard the terminal time T as the additional parameter and to solve a set of the problems of the optimization for various values T. Value T, at which I reaches a minimum, is the solution of the problem. If the solution of the problem can be obtained analytically, the cost functional I may be presented as a function of the parameter T and then one can minimize $I(T)$ with respect to T. Here it is enough to minimize the initial cost function, instead of expanded one since they coincide on the solutions of the problem under consideration.

This approach can be generalized to the case when the value of the variable $x_i(T)$ is not determined at the free terminal time. The problem comes to minimization $I[T, x_1(T)]$ with respect to T and $x_1(T)$.

Returning to the state-variable inequality constrained problems, one can see that consideration of the problem on separate intervals of time with the interior and boundary arcs, can use the aforementioned approach.

The idea of the method offered here for the solving of the multipoint boundary value problem, is in the following. The values $t_i (i = 1, 2, \ldots, N_1)$, and also the values of the unconstrained variables in junction points (here this is $x_1(t_1)$) can be regarded as the parameters of the problem, which must be determined. One obtains their values from the condition of minimum of the functional I, representing it as the function of the introduced parameters. As a result, the optimal control problem is reduced generally to a problem of the nonlinear programming, for which methods of the numerical solving are developed in detail.

The suggested method of reduction of the variational problem to the problem of search of a minimum of a function of several variables is based on the idea, being the main in Ritz's and Stodola's methods. As a matter of fact, this method is closer to the Stodola's method where minimization of a functional is considered with respect to some parameters belonging to expressions for approximating functions.

Based on the shape of the solution of the problem of no constraints, it is possible to suppose that at constraint (8) the solution $x_2(t)$ can have only one constraint arc as it is shown in **Fig 2** (curve 3). In points $t = t_1$ and $t = t_2$, tangential constraints should be satisfied. Constraint (8) is the constraint of the third order of inequality-type. (The constraint is assumed to be of p-th order, as it is known (Bryson and Ho, 1969), when the p-th time-derivative of the constraint is the first to contain the control variable explicitly.) In the problem under consideration, v is the control variable. As a result, the system of tangential constraints in points $t = t_1$ and $t = t_2$ becomes

$$N(z,t) = \begin{bmatrix} x_2 - l \\ x_3 \\ x_4 \end{bmatrix}_{\substack{t=t_1 \\ t=t_2}} = 0. \qquad (9)$$

3. Parameterization of Functional

The equations (9) form the set of the boundary conditions in the interior points of the interval $t \in [0, T]$. Let t_1, t_2 and $x_2(t_1) = a$ be the parameters of the problem, which must be determined. The original two-point boundary value problem corresponding to the state-variable inequality constrained problem under consideration becomes a four-point problem, because also the switching points have to be determined. The solution $x_2(t)$ may be found in the class of continuous functions having constant value equal l on a finite interval of time $t \in [t_1, t_2]$ where $0 < t_1 \le t_2 < T$.

Boundary conditions become

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0; \;\; x_1(t_1) = a; \;\; x_2(t_1) = l; \;\; x_3(t_1) = x_4(t_1) = 0;$$
$$x_1(t_2) = a + l(t_2 - t_1); \;\; x_2(t_2) = l; \;\; x_3(t_2) = x_4(t_2) = 0;$$
$$x_1(T) = \Phi; \;\; x_2(T) = x_3(T) = x_4(T) = 0. \qquad (10)$$

Here a, t_1, t_2 are parameters, which must be determined.

In such a way one can come to the solution of the two boundary value problem for the set of equations

$$\dot{x}_1 = x_2; \;\; \dot{x}_2 = x_3; \;\; \dot{x}_3 = x_4; \;\; \dot{x}_4 = -\lambda_4; \;\; \dot{\lambda}_1 = 0; \;\; \dot{\lambda}_2 = -\lambda_1; \;\; \dot{\lambda}_3 = -\lambda_2; \;\; \dot{\lambda}_4 = -\lambda_3. \qquad (11)$$

Here $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the costate variables.

At that

$$v = -\lambda_4 \tag{12}$$

The solution of the boundary value problem for system (11) in the interval $t \in [0; t_1]$ allows obtaining

$$\lambda_4 = C_7 t^3 / 6 + C_6 t^2 / 2 + C_5 t + C_4.$$

The integration constants expressed through parameters of the problem look like

$$C_4 = 4!5t_1^{-4}(7a - 3t_1);\quad C_5 = 5!3t_1^{-5}(13t_1 - 28a);$$

$$C_6 = 6!2t_1^{-6}(35a - 17t_1);\quad C_7 = 7!10t_1^{-7}(lt_1 - 2a).$$

As a result, the functional in the interval $t \in [0; t_1]$ can be written as the function of required parameters:

$$I_{01} = 6!2t_1^{-7}(9l^2 t_1^2 - 35alt_1 + 35a^2). \tag{13}$$

In a similar manner, $v = -\sum_{k=4}^{7} D_k \dfrac{t^{*(k-4)}}{(k-4)!}$ in the interval $t \in [t_2; T]$, where $t^* = t - t_2$;

$$D_4 = 4!5(T - t_2)^{-4}[-l(4T + 3t_2 - 7t_1) + 7\Phi - 7a];$$

$$D_5 = 5!3(T - t_2)^{-5}[l(15T + 13t_2 - 28t_1) - 36\Phi + 28a];$$

$$D_6 = 6!2(T - t_2)^{-6}[-l(18T + 17t_2 - 35t_1) + 35\Phi - 35a];$$

$$D_7 = 7!10(T - t_2)^{-7}[l(T + t_2 - 2t_1) - 2\Phi + 2a].$$

As a result, the functional can be written in this interval as a function of unknown and given parameters:

$$\begin{aligned} I_{2f} &= 6!2(T - t_2)^{-7}\{9l^2(T - t_2)^2 + 35[\Phi - a - l(t_2 - t_1)]^2 \\ &\quad -35l[\Phi - a - l(t_2 - t_1)](T - t_2)\}. \end{aligned} \tag{14}$$

In the mean interval

$$I_2 \equiv 0. \tag{15}$$

Hence, the total functional (2) can be written as the function of the unknown and given parameters of the problem:

$$I_{0f} = I_{01}(a, t_1, l, T) + I_{2f}(a, t_1, t_2, l, T, \Phi). \tag{16}$$

Violation of the constraint (7) can occur only in intervals $t \in (0, t_1)$ and $t \in (t_2, T)$. Let us consider behavior of this solution in the interval $t \in (0, t_1)$. The possible form of the solution $x_2(t)$ is shown in **Fig 3**. The geometrical sense of violation of the constraint results in the occurrence of the roots of the polynomial $\dot{x}_2(t) = x_3(t) = 0$ in the interval $t \in (0, t_1)$.

Obviously, lack of such roots is the sufficient condition of satisfaction of the constraint (8). Expression for $x_3(t)$ is a polynomial of the fifth order.

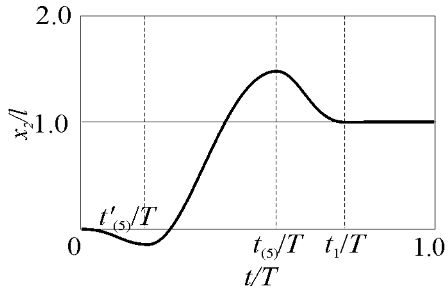

Fig. 3. Possible types of constraint violation

After obtaining of the analytical solution of the boundary value problems (10), (11) and after simplification, this expression can be written as follows:

$$x_3(t) = 60(t - t_1)^2 t^2 (l(7t - 3t_1)t_1 + 7a(-2t + t_1))/t_1^7.$$

Obviously, only one of its roots, namely $t_{(5)} = \dfrac{7at_1 - 3lt_1^2}{7(2a - lt_1)}$, can arise under certain conditions in the indicated interval. The roots are absent, if one of the conditions is satisfied: $t_{(5)} \le 0$ or $t_{(5)} \ge t_1$. Let $t_{(5)} = \dfrac{7at_1 - 3lt_1^2}{7(2a - lt_1)} \ge t_1$. From here it is easy to obtain the constraint for parameters in the form $t_{(5)} \ge t_1 \quad \forall a \in (1/2lt_1, 4/7lt_1]$ or

$$a > 1/2\,lt_1, \quad a \le 4/7\,lt_1. \tag{17}$$

In the same way one can obtain the condition $t_{(5)} \le 0 \quad \forall a \in [3/7lt_1, 1/2lt_1)$ or

$$a \ge 3/7\,lt_1, \quad a < 1/2\,lt_1. \tag{18}$$

Analogously one can obtain the conditions for interval $t \in (t_2, T)$. For lack of violation of the constraint (7) in this interval, the following conditions should be satisfied:

$$\Phi - a - l(t_2 - t_1) > 1/2\,l(T - t_2), \quad \Phi - a - l(t_2 - t_1) \le 4/7\,l(T - t_2) \tag{19}$$

or

$$\Phi - a - l(t_2 - t_1) \ge 3/7\,l(T - t_2), \quad \Phi - a - l(t_2 - t_1) < 1/2\,l(T - t_2). \tag{20}$$

As a result, if the constraint for $x_2(t)$ is active during an interval of positive length, the problem of optimal control can be reduced to the problem of nonlinear programming. Its solution can be obtained only numerically. One can use for this, for example, one of variety of the methods of penalty functions. In the general case, it is necessary to write the extended functional on the basis of the expressions (16), (17) or (18), (19) or (20). Nevertheless, at first it is expedient to investigate the initial parameterized functional (16) on a local extremum. Necessary conditions of the extremum of this function $\frac{\partial I_{0f}}{\partial t_1}=0, \quad \frac{\partial I_{0f}}{\partial t_2}=0, \quad \frac{\partial I_{0f}}{\partial a}=0$ allow obtaining the following solution for the parameters:

$$a = 2/3(l\ T - F), \quad t_1 = 7(l\ T - F)/(6\ l), \quad t_2 = T - t_1, \tag{21}$$

From this $a = 4/7\,lt_1, \quad \Phi - a - l(t_2 - t_1) = 4/7\,l(T - t_2)$, i.e. the extremum point lies on boundary of the constraints (17) and (19). The constraint (7) is not violated at that. As a result, on the basis of the solution (21) it is easy to obtain the analytical solution of the problem for the considered case. It is represented inexpedient to obtain the numerical solution of the problem taking into account constraints (18) and (20) since the found point of the conditional extremum can not give the solution better than the solution obtained on the basis of unconditional minimization of the function (16) without violation of the constraint (7). Numerical checks in ranges of values of the parameters $0 < t_1 \le T/2; \quad T/2 \le t_2 < T; \quad 0 < a \le l\,t_1$ have confirmed it.

So, the found stationary point of the functional determines its global minimum.

4. The analytical solution of the problem

After determination of values (21) of the required parameters, it is possible to obtain the analytical solution of the problem of the optimal control (1) - (3) with the constraint (7). The analytical solution for the phase variable (angle of rotation) for all three intervals of time, and the values of the minimized functional in each interval may be written as follows:

$$t \in [0; t_1]; \quad x_1 = l^6 t^6 m_5 - 3l^5 t^5 m_4 + 2{,}5 l^4 t^4 m_3; \quad I_1 = 11520 l^2 T^{-5};$$

$$t \in [t_1; t_2]; \quad x_1 = \Phi - lT + lt; \quad I = 0;$$

$$t \in [t_2; T]; \quad x_1 = l^6 (t - t_2)^6 m_5 + 3l^5 (t - t_2)^5 m_4 - 2{,}5 l^4 (T - t_2)^4 m_3 + l(t - t_2) + 2\Phi - lT/2;$$

$$I_2 = 11520 l^2 T^{-5}. \tag{22}$$

Here $m_i = (lT - \Phi)^{(-i)}$.

Remaining phase variables and control function v can be obtained as a result of differentiation of these solutions according to the system (1). In **Fig 4**, behavior of the phase variables x_1, x_2, x_3 is shown when the constraint for $x_2(t)$ is active in the interval of finite length.

Now the areas of accessibility of the control for the studied cases may be considered. As it is shown above, the top value Φ_1, which can be reached in the case $x_2(t)_{\max} \to l$, can be obtained from the solution of the problem without constraints: $\Phi_1 = \frac{16}{35} T\, l$.

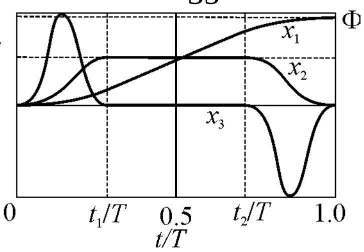

Fig. 4. Behavior of phase variables x_1, x_2, x_3

When the constraint for $x_2(t)$ is active in the interval of finite length,

$$\Phi = l(T - 6/7\, t_1). \tag{23}$$

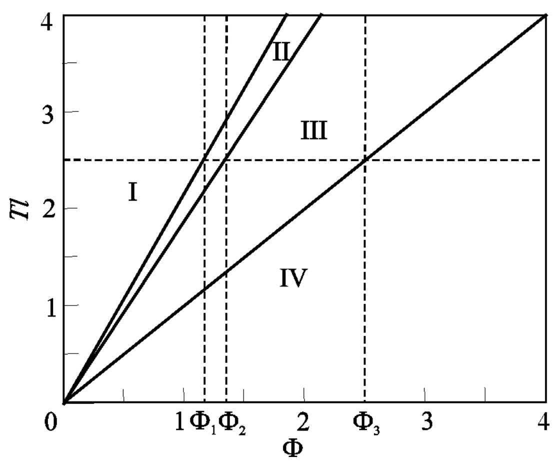

Fig. 5. Areas of accessibility of control

When length of this interval decreases to zero the maximum value of the angle of slewing is determined by the expression $\Phi_3 = 4/7\, T\, l.$

At last, in the extreme case when $x_2(t) \equiv l \;\forall t \in [0, T]$ and which, generally speaking, can not be realized, $\Phi_3 = T\, l.$

In **Fig 5**, the areas of accessibility for the considered cases are shown. The field I can be reached without activization of the constraint (8), in the area II $x_{2\max} = l$ in point $t_1 = t_2 = 0{,}5T.$ According, as the value $t_1 = t_2 = 0{,}5T$ approach to zero Φ is increased from Φ_1 to Φ_2 in this area. Points of the area III can be reached only when the constraint for $x_2(t)$ is active in the interval of finite length.

Values $\Phi / Tl > 1$ (area IV) cannot be reached without violations of the constraint (8).

The solution of the problem of optimal control under consideration for $\Phi \in [\Phi_1, \Phi_2]$ cannot be obtained from the solution (22). On this interval the inequality constraint (8) should be replaced by the boundary condition in the interior point $t_1 = T/2$ of the trajectory. This boundary condition is $x_2(t_1) = l$.

Studying the behavior of the solution over plane $(\Phi, T\,l)$ has shown that the phase variable $x_2(t)$ satisfies the constraint in the point $t_1 = T/2$ on the straight lines $\Phi_{\max} = (16/35)T\,l$ and $\Phi_3 = 4/7\,T\,l$. It is possible to assert at this that such solution $x_2(t)$ corresponds to all points of the interval $\Phi \in [16/35\,T\,l; \quad 4/7\,T\,l]$ also. Really, if to assume that the constraint is not active here, one comes in the contradiction with properties of the solution of the problem with no constraint for which $\Phi_{\max} = (16/35)T\,l.$ If to assume that the constraint is active on a finite interval of time, one comes in the contradiction with property of the solution of the problem with constraint, for which $\Phi_{\min} = 4/7\,T\,l.$

So, one has not choice but to assume that the constraint for $x_2(t)$ in the considered area is active only in one point, otherwise either the solution does not lie in the class of functions with one boundary interval or the constraint (8) is violated.

As a result one comes to statement of the following problem of the optimal control:

$$\dot{x}_1 = x_2; \ \ \dot{x}_2 = x_3; \ \ \dot{x}_3 = x_3; \ \ \dot{x}_4 = v; \ \ I = \frac{1}{2}\int_0^T v^2 dt;$$

$$x_j(0) = 0; \ \ x_1(T) = \Phi; \ \ x_i(T) = 0 \ \ (j = 1,2,3,4; i = 2,3,4); \ \ x_2(t_1) = l. \tag{24}$$

In contrast to the constraints for a phase variable of inequality-type, the constraints of equality-type in an interior point of the interval do not result in the system of tangential conditions.

Here the described above method may be applied in order to solve the problem (24). The values of the phase variables, which are not constrained in the point $t = t_1$ and the value of t_1 may be chosen as the parameters of the problem. As a result, it is possible to present the problem under consideration in the form of two problems with no constraints with fixed boundaries. Respective boundary conditions become

$$x_i(0) = 0 \ \ (i = 1,2,3,4); \ \ x_1(t_1) = a; \ \ x_2(t_1) = l; \ \ x_3(t_1) = c; \ \ x_4(t_1) = d;$$

$$x_1(T) = \Phi; \ \ x_2(T) = x_3(T) = x_4(T) = 0. \tag{25}$$

Reasoning from the aforesaid and also from the fact that the solution is searched in the class of continuous functions, it is possible to state that $x_3(t_1) = c = 0$. In the interval $t \in [0, t_1]$ the solution looks like (4). The integration constants determined from the boundary conditions are

$$C_0 = C_1 = C_2 = C_3 = 0; \;\; C_4 = 4!t_1^{-4}(35a - 15lt_1 - t_1^3 d/6); \;\; C_5 = 5!t_1^{-5}(-84a + 39lt_1 + t_1^3 d/2);$$
$$C_6 = 6!t_1^{-6}(70a - 34lt_1 - t_1^3 d/2); \;\; C_7 = 7!t_1^{-7}(-20a + 10lt_1 + t_1^3 d/6). \qquad (26)$$

In the interval $t \in [t_1, T]$ it is convenient to use the substitution $t^* = t - t_1$. Then the solution looks like (4) with replacement t by t^*. As a result, the problem functional as function of introduced parameters becomes

$$I = \frac{4!5!}{2}\left\{\frac{1}{t_1^7}\left[9l^2t_1^2 - 35alt_1 + 35a^2 + \frac{d^2t^6}{180} + \frac{1}{3}dlt_1^4 - \frac{7}{12}dat_1^3\right] + \right.$$
$$\frac{1}{(T-t_1)^7}[9l^2(T-t_1)^2 - 35(\Phi - a)l(T-t_1) +$$
$$\left. 35(\Phi - a)^2 + \frac{d^2(T-t_1)^6}{180} + \frac{dl(T-t_1)^4}{4} - \frac{7d(\Phi - a)(T-t_1)^3}{12}]\right\} \qquad (27)$$

Parameterization of the solution in the area of accessibility II should be the following stage of the solving of the problem. Violation of the constraint (7) can happen here only in intervals $t \in (0, t_1)$ and $t \in (t_2, T)$. Let us consider the behavior of this solution in interval $t \in (0, t_1)$.

The solution for $x_2(t)$ is shown in **Fig 6** in the area of accessibility II. The geometrical sense of violation of the constraint, as well as in the first case, consists in appearance of roots of the polynomial $\dot{x}_2(t) = x_3(t) = 0$ in the interval $t \in (0, t_1)$. Obviously that lack of such roots is the sufficient condition of satisfying the constraint (8).

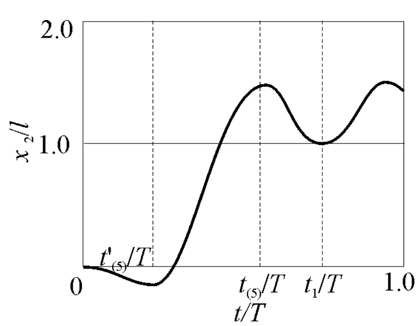

Fig. 6. Possible form of solution $x_2(t)$

Expression for $x_3(t)$ is a polynomial of the fifth extent. After obtaining the analytical solution of the boundary value problem (26), (27) and after simplification, its expression can be written as

$$x_3(t) = t^2(t-1)(t^2(-840\,a + 420\,l\,t_1 + 7\,d\,t_1^3) + t(1260\,a\,t_1 - 600\,l\,t_1^2 - 8\,d\,t_1^4) +$$

$$(-420\,a\,t_1^2 + 180\,l\,t_1^3 + 2\,d\,t_1^5))/t_1^7. \tag{28}$$

Obviously that for satisfying the constraint in the interval $t \in (0, t_1)$ it is sufficiently if the polynomial of the second order

$$t^2(-840a + 420lt_1 + 7dt_1^3) + t(1260at_1 - 600lt_1^2 -$$

$$8dt_1^4) + (-420at_1^2 + 180lt_1^3 + 2dt_1^5) = 0 \tag{29}$$

does not have roots in interior points of this interval. If to assume that it is so, one can obtain the solution of the problem of determination of the extremum of the initial parameterized functional with no constraint.

In this case, the problem can be solved analytically. Necessary conditions of the extremum look like

$$\frac{\partial I}{\partial a} = 35l\left[\frac{1}{(T-t_1)^6} - \frac{1}{t_1^6}\right] + 70\left[\frac{a}{t_1^7} - \frac{\Phi - a}{(T-t_1)^7}\right] + \frac{7d}{12}\left[\frac{1}{(T-t_1)^4} - \frac{1}{t^4}\right] = 0; \tag{30}$$

$$\frac{\partial I}{\partial d} = \frac{d}{90}\left[\frac{1}{t_1} + \frac{1}{(T-t_1)}\right] - \frac{7}{12}\left[\frac{a}{t_1^4} - \frac{\Phi - a}{(T-t_1)^4}\right] + \frac{l}{3}\left[\frac{1}{t_1^3} + \frac{1}{(T-t_1)^3}\right] = 0; \tag{31}$$

$$\ldots\ldots\ldots\frac{\partial I}{\partial t_1} = 45l_2\left[\frac{1}{(T-t_1)^6} - \frac{1}{t_1^6}\right] + 210al\left[\frac{1}{t_1^7} - \frac{1}{(T-t_1)^7}\right] - 245a^2\left[\frac{1}{t_1^8} - \frac{1}{(T-t_1)^8}\right] -$$

$$-\frac{d^2}{180}\left[\frac{1}{t_1^2} - \frac{1}{(T-t_1)^2}\right] - dl\left[\frac{1}{t_1^4} - \frac{1}{(T-t_1)^4}\right] + \frac{28da}{12}\left[\frac{1}{t_1^5} - \frac{1}{(T-t_1)^5}\right] = 0. \tag{32}$$

One of the solutions of the equations (30) – (32) is the stationary point

$$t_1 = \frac{T}{2};\;\; a = \frac{\Phi}{2};\;\; d = \frac{30}{T^3}(7\Phi - 4lT). \tag{33}$$

When $\Phi = -(16/35)Tl, \quad d = -24T^{-2}$. This value coincides with $x_4(T/2)$ for the problem with no constraints at the same Φ value. So, on the boundary of the areas I and II (see **Fig 5**) the solutions of the problem with no constraints for phase variables and the problems with constraint of the equality-type in a point, coincide, values of the minimized functional at this is $I(\Phi_1^-) = I(\Phi_1^+) = 10532\, l^2 T^{-5}$.

As follows from (33), d decreases on absolute value with growth of Φ and at $\Phi = \Phi_2 = T/2$, i.e. on the boundary of the areas II and III, $d = 0$. It follows from this that solutions at $\Phi = \Phi_2^-$ and at $\Phi = \Phi_2^+$ coincide. At this $I(\Phi_2^-) = I(\Phi_2^+) = 23040\, l^2 T^{-5}$.

Let's substitute the solution (33) in the studied polynomial. As a result

$$t^2(-63\,a + 28\,l\,t_1) + t(112\,a\,t_1 - 48\,l\,t_1^2) + (-42\,a\,t_1^2 + 16\,l\,t_1^3) = 0. \qquad (34)$$

It is easy to show from here that at all values $a \in (16/35\, t_1 l, 4/7\, t_1 l)$, which correspond to the area of accessibilities II, $x_2(t)$ has not extremums in the interior points of the interval $t \in (0, t_1)$, i.e. the constraint is not violated.

So, all areas of accessibility are studied and the analytical solutions for the problem of optimal control with no constraints for phase variables, with the equality constraint in a point and with the inequality constraint for phase variables are obtained analytically.

Behavior of $x_2(t)$ over the plane (τ, φ) where $\tau = t/T$, $\varphi = \Phi/(T\, l)$ at $T\, l = 1$ is shown in **Fig 6**.

In the interval $\varphi \in [0; 16/35]$, the solution is obtained on the basis of the solution of the problem of optimal control with no constraints. In the interval $\varphi \in [16/35; 4/7]$, the solution is obtained, as well as in the interval $\varphi \in [4/7; 1]$ by the developed here method. In the interval $\varphi \in [16/35; 1/2]$ $x_2(t)$, the constraint is active only in one point $t_1 = T/2$; in the interval $\varphi \in [4/7; 1]$ $x_2(t)$, the constraint is active in a finite interval of time.

Behavior of the angle of slewing and angular acceleration in time is shown in the following figures (**Fig 9 – 10**). In **Fig 9** the growth ad infinitum of the value of the angular acceleration $x_3(t)$ is well visible when the value φ is approaching to unity. In **Fig 10**, behavior of $x_3(t)$ is shown in the area of accessibility II and in its neighborhood. Here the character of the solution is seen well enough.

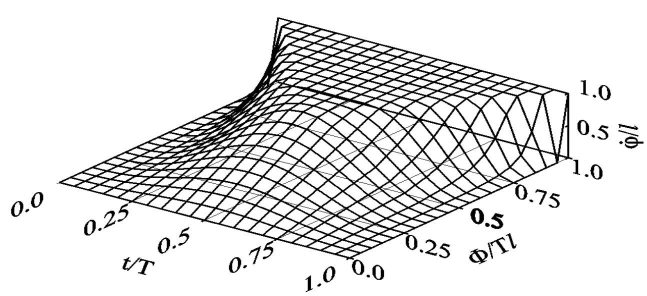

Fig. 7. Behavior $x_2(t)$ over the plane (τ, φ)

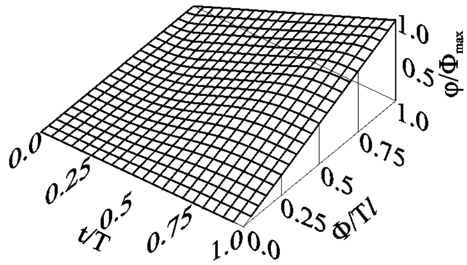

Fig. 8. Behavior of the angle of slewing

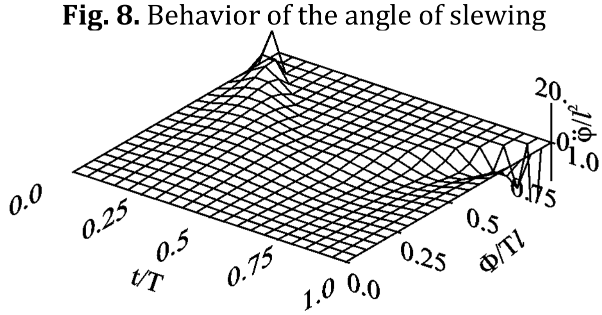

Fig. 9. Behavior of the angular acceleration

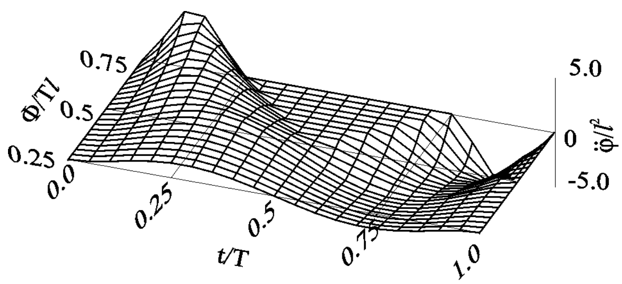

Fig. 10. Behavior $x_3(t)$ in the area of accessibility II and in its neighborhood

Note that since the inequality state constraint is a constraint of 3-rd order jumps are in the costate variables in point $t = t_1$, but this was not illustrate here.

5. Conclusions

The obtained analytical solution can be used as the feed forward control of the slewing both for enough rigid spacecrafts, and for flexible spacecrafts with high-performance system of the feedback control. It can be effectively used as the initial approach at the numerical solving of the boundary value problems of optimal control of slewing of very flexible spacecrafts as it was made by Zakrzhevskii (2008). Besides, this solution can be used as active vibrodamping at shaping the law of deployment of flexible constructions from spacecraft, and also at motion of objects of the ground-based transport with the internal degrees of freedom demanding minimization of relative accelerations of carried bodies.

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