Simulation of Active Vibration Control of a Cantilever Beam using LQR, LQG and $H_{\infty}$ Optimal Controllers

S. M. Khot\textsuperscript{1*} and Yusuf Khan\textsuperscript{1**}

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Abstract

In active vibration control the controller plays an important role in attenuating vibration of the flexible structure. In this paper cantilever smart beam structure is used to study the effect of various optimal controllers for vibration suppression. A single pair of actuator/sensor is placed in collocated form near the root of the beam, being sensor at the bottom surface of the beam. The mathematical modelling is obtained by state space approach. The required Eigen values and Eigen vectors to construct the state space matrices are obtained through the modal analysis using ANSYS\textsuperscript{©}. Three optimal controllers LQR, LQG and $H_{\infty}$ are designed by using state feedback and output feedback law. LQR optimal control gain is calculated by using state feedback and output feedback control law. Considering the process and measurement noises Kalman gain is calculated, and LQG regulator is obtain by combining the LQR gain and Kalman gain. By considering weighting functions close loop of plant is constructed by $H_{\infty}$ controller method. The simulation study of active vibration control is carried using MATLAB\textsuperscript{©} and based on results obtained it can be concluded that $H_{\infty}$ controller has a good close loop dynamic performance than LQR and LQG controller.

Keywords: Flexible structure; ANSYS\textsuperscript{©} software; Active vibration control; LQR controller; LQG controller; $H_{\infty}$ controller

1. Introduction

Vibration in most of the cases is undesirable because it causes unpleasant noise, unwanted stress in structures, and malfunction or failure of system. Passive vibration control methods are not suitable, when the frequencies of the disturbance vary with time and the frequencies of sound and vibration are low. This method also adds increase in weight of structures and unable to adapt with environmental changes. An interesting alternative solution is to actively control vibrations especially by the use of smart materials (Poshtan and Yousefi, 2011). Such materials can act as sensors, which sense the disturbances in the structures, and actuators which are capable of...
applying the controlling force (Preumont, 2011).

Zhang et al. (2010) had carried simulation study of active vibration control of cantilever beam using piezoelectric materials with LQG controller using ANSYS© software environment. The first 4 ranks of modal frequencies and shapes are extracted through modal analysis and then Kalman filter is designed using LQG optimal control method. The dynamic response is analyzed by giving step input to the plant and using LQG controller to minimize the vibration response by minimizing the cost function by state feedback control law and Kalman filter (Poshtan and Yousefi, 2011; Zhang et al., 2010). Zhang et al. (2008) also studied the active vibration control of beam using LQR optimal control theory. With a multiple-input and multiple-output (MIMO) control system, linear quadratic control methods are the preferred choice and can be used effectively for multimode vibration suppression. Linear Quadratic Regulator (LQR) control approach is well suited for the requirements of damping out the effect of disturbances as quickly as possible and maintaining stability robustness. Zhang et al. (2008) studied the active vibration control of beam using LQG and H-∞ optimal controller and compared the results of both controllers. It has been seen that effectiveness comparison of all three optimal controllers is not attempted for the same smart structures.

In the present study an attempt is made to apply three optimal controllers for the same smart structure to control vibration for comparing the performance. The mathematical modelling is carried in MATLAB© by state space approach using the first ten rank of Eigen values and Eigen vectors obtained through modal analysis using ANSYS©. To reduce the computational time, the model is reduced by taking only those modes of frequencies which are most contributing to the overall response on the basis of highest dc gain values. The full and reduced model is used to design optimal controllers such as, LQR, LQG and H-∞ by using state feedback and output feedback law. The simulation study of active vibration control is carried out using these three different controllers and performance is compared.

2. Modal Analysis

A modal analysis gives the undamped natural modes of a system in increasing order of frequency (Rao, 2011). The model of a cantilever beam with the single sensor/actuator is built in ANSYS© software. And through modal analysis 10 ranks of modal frequencies and mode shapes are extracted (Khot et al 2010; 2011). The beam having size of 508 x 25.4 x 0.8 mm³ and sensor/actuator of dimensions 76.2 x 25.4 x 0.305 mm³ are selected for analysis. Single pair of actuator/sensor is placed in collocated form at nearer to the root of the beam, with the sensor at the bottom surface of the beam. The element type chosen for beam is solid 45 and for piezoelectric patch is solid 5. Mesh size of 70 x 4 x 1, is arrived by mesh conversance study. The modal analysis of integrated structure is performed in ANSYS© and Eigen value of first 10 modes frequencies in Hz are given in table 1.

<table>
<thead>
<tr>
<th>Table 1 Eigen values for first ten modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modes</td>
</tr>
<tr>
<td>Eigen values</td>
</tr>
<tr>
<td>Modes</td>
</tr>
<tr>
<td>Eigen values</td>
</tr>
</tbody>
</table>
Input and output is at the tip of the beam, since the only node of interest is the tip node, so the row of the modal matrix corresponding to the tip node may be retained. The Eigen vector corresponding to the tip is,

\[ X_n = [12.51 \ 12.07 \ 11.38 \ 11.81 \ 12.23 \ 12.35 \ 12.25 \ 11.94 \ 11.904 \ 11.578] \]

These Eigen values and Eigen vectors are used to construct state space models in MATLAB©.

### 3. Mathematical Modeling

The mathematical model of cantilever beam is constructed in MATLAB©, with an impulse force as input at the tip, and the tip deflection is output of the system. The state space matrices are constructed by using Eigen values and Eigen vectors obtained from modal analysis using ANSYS©. The input output state equations are

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

Where, \( x \) is the column vector representing the state of the system. \( y \) is the output matrix, \( u \) is the input matrix, \( A \) is the system matrix, \( B \) is the force matrix, \( C \) is the output matrix, \( D \) is the direct transmission matrix (Ogata, 2010).

\[
A = \begin{bmatrix}
0 & 1 & & & & & & & & 0 \\
-\omega_n^2 & -2\zeta\omega_n^2 & & & & & & & & \\
& \ddots & \ddots & \ddots & \ddots & & & & & \\
& & \ddots & \ddots & \ddots & \ddots & & & & \\
& & & \ddots & \ddots & \ddots & \ddots & & & \\
0 & & & & & & & & \omega_n^2 & -2\zeta\omega_n^2
\end{bmatrix}
\]

\[ F_p = X_n \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix} \]  

\[ B = \begin{bmatrix}
0 \\
F_{p_1} \\
\vdots \\
0 \\
F_{p_n}
\end{bmatrix} \]

\[ C = [X_{n_1} \ 0 \ \ldots \ \ X_{n_n} \ 0] \]

\[ D = [0] \]

Where, \( n = \) number of modes.  
\( \zeta \) 1, \( \zeta \) 2, \ldots, \( \zeta \) \( n \) = constant=0.02
F = force in physical coordinate.
Fp = force in principle coordinate (Hatch, 2001).
The system matrices A, B, C and D are used to form state space of system, using ss function in MATLAB®. The transient and frequency responses are plotted in MATLAB® by using lsim and bode function respectively. The model get reduced by taking only those modes of frequencies which are most contributing to the overall response on the basis of highest dc gain values. Transient and frequency response of full and reduced model of open loop system is given in figure 1.

![Fig. 1. Transient and frequency response.](image)

In Fig. 1, the full model result is overlapped with reduced model result to show the differences. For the present study highest contributing five modes are taken to construct reduced model of the system. From Fig. 1 it is clear that, the responses (frequency and transient) of reduce model of the system closely matches with the responses of full model. So instead of full model reduced model may be used to represent the system. The developed state space model is used for designing optimal controllers, which is discussed in next section.

4. Optimal Controllers

In control theory, a controller is a device which monitors and affects the operational conditions of a given dynamical system. The operational conditions are typically referred to as output variables of the system which can be affected by adjusting certain input variables. The objective of optimal control theory is to determine the control signals that will cause a process to satisfy the physical constraints and at the same time minimize (or maximize) some performance criterion. The systematic approach to calculating an optimal control law begins with the choice of a performance index or cost function (Bryson, 2002).

In close loop system with controller, the actuator produce controlling force, this acts on the nodes at the ends of the actuator in the X direction, while the exciting force and output displacement are at the tip of beam along the Z axis. Hence the eigen vectors pertaining to the UX and UZ
displacement are required to construct the Xn matrix. Thus the first few rows of the Xn matrix correspond to the UX displacement of the nodes on one end of the actuator, the next few rows correspond to the UX displacement of the nodes on the other end of the actuator and the last row corresponds to the UZ displacement of the tip node. By using the Eigen values and Xn matrix (Eigen vectors) state space modeling of close loop system is obtained. There are two control laws which can be used to design controller, State feedback and Output feedback.

4.1 State Feedback

4.1.1. Linear Quadratic Regulator (LQR) optimal controller

The LQR controller is designed to determine the optimal controller gain. And the objective is to minimize the following quadratic cost function:

\[ J = \frac{1}{2} \int_0^\infty (x^T Q x + u^T R x) \, dt \]  

(8)

Where \( Q \) and \( R \) are suitably chosen positive semi-definite weighting matrices and \( u \) is the control force to be applied.

In state feedback controller law, the relative magnitude of \( Q \) and \( R \) are selected to trade off requirements on the smallness of the state against requirements on the smallness of the control force. The control force is of the form,

\[ u = -Kx \]  

(9)

Where \( K \) is optimal controller gain, which is,

\[ K = R^{-1} B^T P \]  

(10)

Where \( P \) is the unique symmetric positive semi-definite solution to the following algebraic Riccati equation,

\[ PA + A^T P + C^T C - PBR^{-1}B^T P = 0 \]  

(11)

The LQR controller gain (\( k \)) for state feedback is calculated by using the lqr function in MATLAB. 

>>[K,P,E]=lqr(A,B,Q,R);

In this paper \( Q \) and \( R \) are tacked as 1 and 1e-7 respectively.

The closed-loop dynamics arrived at by substituting state feedback equation in system dynamic equation.

\[ \dot{x} = (A - BK)x + Bu \]  

(12)

The close loop of plant is constructed in MATLAB as,

>>sysCL=ss(A-B*K,B,C,D);

The transient and frequency responses of LQR controller with state feedback law are plotted in MATLAB using lsim and bode function respectively, and it's given in Fig.2 and Fig.3 respectively.
The Fig. 2 and Fig. 3 shows the uncontrolled and controlled transient and frequency responses of tip of the flexible structure. The settling time of tip of the cantilever beam with LQR controller is 3.64 sec.

![Fig. 2. Transient response at state feedback.](image1)

![Fig. 3. Frequency response at state feedback.](image2)

4.1.2. Linear, Quadratic, Gaussian (LQG) optimal controller

Optimal Linear, Quadratic, Gaussian (LQG) compensators is based upon a linear plant, a quadratic objective function and an assumption of white noise that has normal or Gaussian probability distribution. A state-space realization of the optimal compensator for regulating a noisy plant with state-space representation is (Tewari, 2002)
\[ \dot{x} = Ax + Bu + Fv \]  

\[ y = Cx + Du + z \]

where, \( v \) and \( z \) are process noise vector and measurement noise respectively. The design of LQG optimal controller with state feedback is given below.

To design LQG optimal compensator, the regulator gain \( K \) and Kalman filter which filter the noises are required. The regulator gain \( K \) is calculated from equation (10) by solving the algebraic Riccati equation (11) by minimizing the LQR cost function equation (8). Kalman filter is designed for the plant assuming a known control input \( u \), a measured output \( y \), and white noises \( v \) and \( z \), with known power spectral densities. The Kalman filter is designed to provide an optimal estimate of the state-vector \( x \) (Tewari, 2002). Figure 4 shows the block diagram of close loop of plant with LQG optimal controller.

![Block Diagram of LQG Controller](image)

**Fig. 4.** Close loop of plant with LQG optimal controller.

Combination of separately designed optimal regulator and Kalman filter give optimal compensator, which generates the input vector \( u \), based upon the estimated state-vector \( x \), rather than the actual state-vector \( x \), and the measured output vector \( y \) (Tewari, 2002). A state-space realization of the optimal compensator for regulating a noisy plant with state-space representation of equations (13) and (14) is given by the following state and output equations,

\[ \dot{x} = (A - BK - LC + LDK)x + Ly \]

\[ u = -Kx \]

Where, \( K \) and \( L \) are the LQR optimal regulator gain and Kalman filter gain matrixes respectively.

The Kalman gain \( L \) is calculated in MATLAB© by using kalman or lqe function.

```matlab
>> [est,L,E]=kalman(sysp,v,z);
>> [L,P,E]=lqe(A,B,C,v,z);
```
Where, sysp is state space of plant, v and z are the noises. In the present study v and z are tacked as ρBBᵀ and CᵀC respectively, and ρ is a constant number.

The LQG optimal regulator which is combination of K and L is calculated in MATLAB© by using reg function,

```matlab
>> lqgreg = reg(sysp, K, L);
```

The close loop of plant is constructed in MATLAB© as,

```matlab
>> sysCL = feedback(lqgreg, sysp);
```

The transient and frequency response of LQG controller with state feedback law are given in Fig. 5 and Fig. 6 respectively.

**Fig. 5.** Transient response at state feedback.

**Fig. 6.** Frequency response at state feedback.
The settling time in LQG controller is 3.06s, which is lesser as compared to LQR controller, because it filters the white noises of the system. Also, it can be derived from the frequency responses that in case of LQR controller, the amplitudes follow the pattern of the amplitudes of uncontrolled frequency response while in case of LQG controller; the amplitudes have considerably died down thus resulting in much better vibration control.

4.1.3. $H_{\infty}$ controller

The LQG controller developed by using optimal regulator (LQR) & Kalman filter exhibit good performance but there robustness to process and measurement of noise cannot be guaranteed. The $H_{\infty}$ optimal control design technique, however, directly address the problem of robustness by deriving controllers which maintain system response and error signals to within prescribed tolerances, despite the presence of noise in the system (Tewari, 2002). Figure 7 shows a plant with transfer matrix $G(s)$, input vector $U(s)$, and output vector $Y(s)$, being controlled by a feedback compensator with transfer matrix $K(s)$.

![Fig. 7. Close loop control system](image)

The vector $w(s)$ contains all inputs external to the closed-loop system, i.e. process and measurement of noise vectors, as well as the desired output vector. The vector $z(s)$ contains all the errors that determine the behaviour of the closed-loop system, i.e. the estimation error and the tracking error vectors.

![Fig. 8. Close loop plant with weights](image)

Fig.8 shows the block diagram of close loop plant with noises. Where, $G$ is the transfer function of the plant. The regulator gain $K$ is calculated in MATLAB© by using hinfopt function:

```matlab
>>[K,sysCL,GAM]=hinfsys(P);
```

where,
\[ P = \text{AUGW}(G, w_1, w_2, w_3); \]

\[ w_1 = \frac{8 \times 10^{-5}(s^2 + 100s + 1500)}{0.01(s^2 + 2s + 1)} \]  \hspace{1cm} (17)

\[ w_2 = \text{constant} \]  \hspace{1cm} (18)

\[ w_3 = [] \]  \hspace{1cm} (19)

\[ [\text{GAM, acp, bcp, ccp, dcp, acl, bcl, ccl, dcl}] = \text{hinfopt}(P); \]

The close loop of plant with H-\(\infty\) optimal controller is constructed in MATLAB\(\text{©}\) as,

\[ [\text{sysCL} = \text{ss}(\text{acl, bcl, ccl, dcl}); \]

The transient and frequency responses of close loop plant are given in figure 9 and 10 respectively.

**Fig. 9.** Transient response at state feedback.

**Fig. 10.** Frequency response at state feedback.
From the Fig. 9 and Fig.10 it can be seen the stable close loop response of system and it is clear that all the peaks in frequency response get eliminated. The settling time with H-∞ controller is 2sec. From the transient response it is clear that the H-∞ controller has a good close loop dynamic performance than LQR and LQG controller.

4.2 Out-put Feed-back

4.2.1 Linear Quadratic Regulator (LQR) optimal controller

In output feedback controller law, the output, y(t) is used, rather than the state-vector, x(t), which is included in the objective function for minimization. The reason for this may be either a lack of physical understanding of some state variables, which makes it difficult to assign weightage to them, or that the desired performance objectives are better specified in terms of the measured output (Tewari, 2002). The algebraic Riccati equation for LQR output feedback law is

\[ PA + A^T P - PB R^{-1}B^T P + [Q - SR^{-1}S] \]

Where, \([Q - SR^{-1}S]\) is a positive semi-definite matrix.

However, MATLAB© provides the function lqry for solving the output weighted linear, quadratic optimal control problem, which only needs the plant coefficient matrices A, B, C, and D, and the output and control weighting matrices, Q and R, respectively, as follows,

\[
\begin{align*}
&\text{>>[K,P,E]=lqry(A,B,C,D,Q,R);} \\
&\text{The close loop of plant is constructed in MATLAB© as,} \\
&\text{>>sysCL=ss(A-B×K,B,C,D);} \\
&\text{The transient and frequency responses of LQR controller with output feedback law are given in Fig.11 and Fig.12 respectively.}
\end{align*}
\]

Fig. 11. Transient response at output feedback.
It can be seen from the transient responses of state and output feedback, that the settling times in both the cases are 3.64sec and 3.86sec respectively. The only advantage of using output feedback is that it applies control gain directly to sensor output. Hence no estimation of state variables is involved in this approach, and it simplifies the internal complexity of controller. This leads to the poor stability of system response.

4.2.2 Linear, Quadratic, Gaussian (LQG) optimal controller
For LQG controller with output feedback, the regulator gain is calculated by using LQR output controller law as explained earlier. C output matrix become C×A and D direct transmission matrix become C×B, and it is used to construct state space model of plant. And remaining design procedure of LQG controller with output feedback is same as for LQG controller with state feedback. The transient and frequency response of LQG controller with output feedback law are given in Fig.13 and Fig.14 respectively.

It can be seen from the transient responses of state and output feedback, that the settling times in both the cases are 3.06s and 2.82s respectively. Generally, the output weighted regulator produces a much smoother response, which decays faster than that of the corresponding LQRY regulator. The application of output weighted optimal control is not limited to flexible structures, and can be extended to any plant where smoothening of the transient response is critical.

4.2.3 $H_{\infty}$ controller

To find a constant output-feedback gain $K$ as described, one may define the value functional given by Eq.

$$J(K, d) = \int_0^\infty (x^T Q x + u^T R u - \gamma^2 d^T d) dt$$

(21)

$$= \int_0^\infty [x^T (Q + C^T K^T R K C) x - \gamma^2 d^T d] dt$$

(22)

With, $KC = R^{-1}(B^T P + L)$

Where, $P>0, P^T=P$ is a solution of following algebraic Riccati equation,

$$PA + A^T P + Q + \frac{1}{\gamma^2} P DD^T P - PBR^{-1} B^T P + L^T R^{-1} L = 0$$

(23)
The regulator gain $K$ is calculated in MATLAB® by using hinfsyn or missyn function,

\[
>>[K,sysCL,GAM]=hinfsyn(P);
\]

\[
>>[K,sysCL,GAM]=missyn(G,w1,w2,w3);
\]

The close loop of plant with $H$-∞ optimal controller is constructed in MATLAB® as,

\[
>>\text{sys Ctrl}=\text{ltf}(P,K);
\]

The transient and frequency response of $H$-∞ controller with output feedback law are given in Fig.15 and Fig.16 respectively.

![Fig. 15. Transient response at output feedback](image1.png)

![Fig. 16. Frequency response at output feedback](image2.png)
From the transient response it is clear that the $H^-\infty$ controller has a good close loop dynamic performance than LQR and LQG controller, because the structure with $H^-\infty$ controller has less settling time i.e 1.96sec as compare to LQR and LQG.

The comparison of performance of all three optimal controllers by state feedback and output feedback are presented in next section.

5. Results

The results of LQR, LQG and $H^-\infty$ optimal controllers are overlapped in order to compare their performance, in terms of transient and frequency responses. The comparisons of results of these three controllers by state feedback law and output feedback law are presented here.

5.1 Comparison of Optimal Controllers by State Feed-back

The response of LQR, LQG and $H^-\infty$ optimal controllers by state feedback in term of transient and frequency response are given in Fig. 17 and Fig. 18 respectively.

![Fig. 17. Transient response at state feedback.](image)

![Fig. 18. Frequency response at state feedback.](image)
5.2 Comparison of Optimal Controllers by Output Feed-back

The response of LQR, LQG and $H_{\infty}$ optimal controllers by output feedback in term of transient and frequency response are given in figure 17 and 18 respectively.

In Fig. 17 to Fig. 20 the transient and frequency responses of three controllers are compared. From the transient response it is clear that the $H_{\infty}$ controller has a good close loop dynamic performance than LQR and LQG controller. It has been seen that in frequency response of three controllers, the amplitudes of LQR controller follow the pattern of the amplitudes of uncontrolled frequency response, the amplitude of LQG controller have considerably died down, while in case of $H_{\infty}$ controller amplitude get constant at higher frequencies. The comparison of all three
controllers with state feed-back law and out-put feed-back in terms of settling time are given in table 2.

Table 2 Comparison of settling time of various optimal controllers

<table>
<thead>
<tr>
<th>Control laws</th>
<th>Optimal controller</th>
<th>Settling time</th>
</tr>
</thead>
<tbody>
<tr>
<td>State feed-back</td>
<td>LQR</td>
<td>3.64s</td>
</tr>
<tr>
<td></td>
<td>LQG</td>
<td>3.06s</td>
</tr>
<tr>
<td></td>
<td>H-∞</td>
<td>2.00s</td>
</tr>
<tr>
<td>Out-put feed-back</td>
<td>LQR</td>
<td>3.86s</td>
</tr>
<tr>
<td></td>
<td>LQG</td>
<td>2.82s</td>
</tr>
<tr>
<td></td>
<td>H-∞</td>
<td>1.96s</td>
</tr>
</tbody>
</table>

6. Conclusion

Active vibration control using three optimal controllers, such as LQR, LQG and H-∞ with state feedback and output feedback law are investigated in the present study. The Eigen values and Eigen vector is imported in MATLAB© and state space model is built. The state space model for full and reduced is constructed, and it is used to analyse for vibration response. The full and reduced model is used to design optimal controllers such as, LQR, LQG and H-∞ with state feedback and output feedback law. The simulated results of all the three controller is compared and it is concluded that the H-∞ controller has a good close loop dynamic performance than LQR and LQG controller. H-∞ controller has better response than other controller because it is designed for the worst situation along with the required performance and robustness objective.

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